Chiral Lagrangian theories of Gravitation

The most resilient effective theory of gravity to date is Einstein’s General theory. Reformulations of Einstein’s field equations over the past forty years have been stimulated by influences such as the developments in the study of gauge theories, the construction of half-flat solutions in the 1980’s by, for example, Penrose, Newman and Plebanski and the recasting of the Hamiltonian formulation of general relativity in terms of new variables by Ashtekar. The latter, itself a response to the first two influences, reintroduced the idea of regarding the connection and bases of two forms as primary dynamical variables with the metric a secondary derived variable. The novelty of such schemes has been the focus on so-called complex, \( \mathbb{C} \)-valued chiral actions and complex versions of Einstein’s equations.

Einstein’s theory seems best re-formulated as an Action Principle based on two dynamical variables, the metric and space-time connection. Many of its latter-day variants are well motivated, trying say, to make their Classical formulation more amenable to quantization programs such as the path integral approach or by more fully encapsulating Mach’s principle of inertia. The Darwinian sieve is filtering out the still fittest of these formulations to the vastly increasing astronomical dataset of late.

Tetrad falling frame formulation of General theory

Einstein’s freely falling “elevator” frame, upon which his Equivalence Principle (of inertial and gravitational masses) is founded, is cogently encoded in the “verbein”, “tetrad” or its co-frame formulation of Ellie Cartan. This “Gauge” formulation mirrors Yang-Mill’s theory of the three forces of the Standard model. Einstein’s metric is derived from the Euler-Lagrange equations of Action principle in which the one-form or its bi-vector big sister are the dynamical” quantizable fields in the path integral. These dynamical fields are afforded ”particle” status as the mediating ”graviton”.

Of the Galileon-like theories that abound, dRGT ”massive gravity” imbues “inertial mass” to the spin-2 particle - graviton, the excitation of a quantum field that gives rise to the otherwise long range gravitational force mediator. Being massive, this ”graviton” possesses five polarization (0,1,2) ”chiral” handedness states compared to the two (2) ”helicities” of its massless, twisted svelter Einsteinian brother.

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Lagrangian of Chiral left-handed Dynamical Field

To this end, Plebanski’s formulation in the mid 1970’s of a chiral first order variational principle for general relativity in which the basic field variables are sl(2, C) valued two-form, connection form and a spinor-valued zero-form is of interest here,

\[
i \frac{i}{2} \mathcal{L}_\Sigma(\Sigma, \Psi, \Gamma) = \{\Sigma^{AB} \wedge F_{AB} - \frac{1}{2} \Psi_{ABCD} \Sigma^{AB} \wedge \Sigma^{CD}\}.
\]

The latter term, \(-\frac{1}{2} \Psi_{ABCD} \Sigma^{AB} \wedge \Sigma^{CD}\) forces the Ricci part of the Curvature two-form to vanish and crucially the constraint arising from the variation of \(\Psi\) dictates that \(\Sigma^{AB}\) is determined by a tetrad, according up to \(SL(2, C)\) transformations on primed indices. This Lagrangian is incomplete for real General Relativity in that reality conditions on the \(SL(2, C)\) valued two forms, \(\Sigma^{AB}\) need to be put in by hand, to ensure a real Lorentzian space-time, \(\Sigma^{AB} \wedge \Sigma^{A'B'} = 0\), and \(\Sigma^{AB} \wedge \Sigma_{AB} + \Sigma^{A'B'} \wedge \Sigma_{A'B'} = 0\). The "chiral" nature of the formulation is in the sense that local Lorentz representations involve only \(SL(2, C)\) and not its conjugate, \(\overline{SL(2, C)}\). That is the dynamical field object rather than being a mixed index co-frame, \(\theta^{AA'} = \theta^{AA'}\mu dx^\mu\), a Hermitian matrix-valued one-form from which the (real Lorentzian) metric is given as \(ds^2 = \epsilon_{AB} \epsilon_{A'B'} \theta^{AA'} \otimes \theta^{BB'}\) it is rather defined a priori as the anti-self dual two-form, \(\Sigma^{AB} := \frac{1}{2} \theta^{A}_A \wedge \theta^{B}_{B'}\).

Symplectic Metric of Spin space and Soldering form

The view here is that the symplectic metric \(\epsilon_{AB}\) of spin space is fixed once and for all and that the soldering form, \(\sigma^{AA'}\mu\) contains all the information pertaining to the metric \(g_{\mu\nu}\). For real general relativity the soldering functor is required to be real, \(\bar{\theta}^{AA'} = \theta^{AA'}\). In essence, the on-shell Euler-Lagrange equations for the Chiral Plebankski action enable space-time, \(M\) to effectively admit the usual global null tetrad: a spinor-structure \(PB\), a principal fibre bundle with structure group \(SL(2, C)\), the gauge group for spinor dyads.

In quantised versions of classical field theories, massless gauge vector bosons have spins orientated in the same direction along their (parallel to the) axis of motion regardless of the observer’s viewpoint. The chirality of these “force mediators” is absolute due to both the invariance and finite speed of light: as massless particles moving at the speed of light no massive (slower) observer can travel in a faster reference frame so that the boson would appear to be going in a reverse (anti-parallel) relative direction. With a massive graviton this is no longer the case. Thus its extra modes are attributed to its handedness that is now ambiguous in that NOT all (massive) observers (who after all are the only ones who can note the tick of time) would see the same chirality. Such even-handed chirality is called helicity. Now whereas the direction of spin of massless particles are not affected by a Lorentz boost (the relativistic equivalent of a Galilean change of reference frame) in the direction of motion of the particle, the sign of the projection for a dRGT graviton is not fixed for all reference frames and its helicity has orientation.
Chiral Variational Principle for Coupled Graviton-Photon-charged Higgs-Fermionic field

Viewed as a ‘constrained’ BF theory Plebanski’s formulation lends itself to natural generalisations as in the Einstein-Maxwell theory of Robinson that employs gauge group of $gl(2,\mathbb{C}) = sl(2,\mathbb{C}) \oplus \mathbb{C}$ field valued variables, the $GL(2,\mathbb{C})$-valued $S^A B$-forms being the primary field variable determined up to $GL(2,\mathbb{C})$ transformations on primed indices. With a $GL(2,\mathbb{C})$-valued connection, $\gamma^A B$ the chiral Lagrangian for Einstein-Maxwell reads

$$\frac{i}{2} \mathcal{L}_S(S, \alpha, \gamma) = f^A B \wedge S^B A + \frac{1}{2} \alpha_B^{A D} C S^B A \wedge S^D C.$$

The theory is constrained in the sense that the Lagrange multiplier term, $\alpha$ is needed (spoiling the precise BF form of the Lagrangian) to ensure that the basic two form field variable is derived from a co-frame. Such chiral formulations lead to complex vacuum field equations for a complex metric with the real theory recovered only upon imposing (by hand) reality conditions.

Work on chiral variational principles was extended to include various matter fields by Capovilla et al and Pillin and following their prescriptions, the effects of coupling (charged) Higgs and fermionic matter fields to a $GL(2,\mathbb{C})$ formulation of the photon and graviton requires the restriction of the gauge group to be $SL(2,\mathbb{C}) \otimes U(1)$ in order to recover a real theory. Such a formulation is distinct from those theories which use a linear connection to incorporate Maxwell and deploys a spinor density to describe the charged complex (Higgs) scalar field in a formulation developed by Plebanski in unpublished notes.

References