# Chiral BF-like theory of Gravito-electromagnetic waves

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**Abstract**

Plebanski's Modified Chiral BF Lagrangian for General Relativity [5] possesses local $SL(2,\mathbb{C})$ Lorentz representations, whose internal indices $AA'$ associated to the spin structure become spinor indices through a dynamical soldering bivector 2-form, $\Sigma^{AB}{}_{\mu\nu}$. The spin structure group is the gauge group for spinor dyads, a six parameter Lie subgroup of $GL(2,\mathbb{C})$. By employing this full 8-dimensional group structure, using Robinson's [14] first order variational formalism comprising connection and handed self-dual bivector form, $S^A{}_B$, (an oriented pseudo-tensor density) charged fermion and scalar fields are coupled to a unified gravitational-electromagnetic field. When off-shell reality conditions on the fermion fields are imposed a restricted gauge $SL(2,\mathbb{C})\otimes U(1)$ symmetry results. Unlike in more recent work of Smolin et al [38] the construction remains chiral and such "simplicity" conditions are kept off shell. The ambiguity in the definition of the metric described by the restricted orthogonal group mapped to $GL(2,\mathbb{C})$ is made explicit through its transformation properties as developed in unpublished notes of Peblanski, [6]. The symplectic metric $\tilde{\epsilon}_{AB}$ of the spin bundle "density" is viewed as primary and the soldering functor, $\tilde{\sigma}^{AA'}{}_\mu$ embodying metricity $g_{\mu\nu}$ on space-time is derived from field equations of a Chiral Einstein-Maxwell Lagrangian. Charged scalar fields are constructed as complex valued weighted-spinor densities to be interpreted as Higgs or Brans-Dicke dilaton in nature. Gauge fixing and Integrability issues of the $GL(2,\mathbb{C})$ unified-field potential are discussed by making contact with Lanczos potential and Bel-Robinson Entropy super tensors for the weak field equivalent of a Weyl-Faraday curvature spinor of Penroses Cyclic Conformal Cosmology radiation-only endgame.

***Keywords:*** Chiral, Modified BF, scalar density, Lanczos Potential, Bel-Robinson

# Preface

According to the Cosmic Cyclic Cosmology, CCC of Penrose the remnants of a once material universe will be massless photons riding a gravitational wave from the last pop of Black hole Hawking radiation. Tracing out a timeless null geodesic this final emanation marks the expunging of the ticking of time. With purgatory tripped we have arrived at oblivion, the non-era pervaded by gravito-electromagnetic radiation,

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a scale-less conformal space, that becomes the precept for material renewal. The Cyclicality of Penrose renders the end game, a beginning sourced by lone immaterial Weyl-Curvature and Faraday Fields that are not Peeling off to infinity after all.

# Motivating Variants of Einstein's Gravitation

Reformulations of Einstein's field equations, GR over the past sixty years have been stimulated by influences such as the developments in the study of gauge theories, the construction of half-flat solutions in the 1980's by, for example, Penrose, Newman and Plebanski and the recasting of the Hamiltonian formulation of general relativity in terms of new variables by Ashtekar. The latter, itself a response to the first two influences, reintroduced the idea of regarding the connection and bases of two forms as primary dynamical variables with the metric a secondary derived variable. The novelty of such schemes has been the focus on so-called complex, $\mathbb{C}$-valued chiral actions and complex versions of Einstein's equations. Other motivations are in trying to make classical formulations more amenable to quantization programs such as the path integral approach.

## Metric Emergence

That the metric is merely a derived unobservable gauge quantity motivates geometric algebra programs that posit either the pre-eminence of conformal equivalence of rays or spin bundle structure over a derived space-time. In Penrose's twistor program, a twistor, $Z^\alpha = (\rho^A, \pi_{A'})$ represents the enitre (ray) history of a free classical massless particle with definite helicity, s from which space-time emerges as the common points of intersection of ray families. Locally at least, two valent anti-symmetric twistors called "infinity twistors" fix the (derived/implied) metric structure of space-time. The twistor algebra viewed as an extended spinor algebra is then to be viewed as the fundamental structure for unifying quantum mechanics and general relativity. Equivalence is driven by the symplectic spin group, $Spin(1,3)$ of unitary determinant, a 2-1 map of the group of orthogonal transformations, $SO(1,3)$[1] and in particular its subgroup, $Spin^+(1,3)$ a double-cover of the restricted[2] orthogonal group, $SO^+(1,3)$. The latter in turn has universal covering group $SL(2,\mathbb{C})$ that facilitates represention of null vectors through the outer product of left and right-handed Weyl spinors[3],

$$so(1,3\mathbb{C})^+ \oplus so(1,3\mathbb{C})^- \xrightarrow{\sim} \overline{sl(2,\mathbb{C})} \oplus sl(2,\mathbb{C}). \tag{1}$$

Geometrically Einstein's freely falling elevator is captured in the soldering functor ${\sigma^a}_\mu$ providing the local identification of space-time with local Lorentz quantities of the observer within the definition of the non exact differential form, $\theta^a = {\sigma^a}_\mu dx^\mu$, it cogently encodes his Equivalence Principle of inertial and gravitational masses. These co-frames being not integrable at each event constitute a non-coordinate (anholonomic) Lorentz basis form the Cartan G-structure with cotangent bundle, T*B soldered to

[1] Conformal transformations preserving angles define a group $C(0,2)$ double-cover rep. of $SO(1,3)$

[2] that subgroup of transformations continuously connected with the identity

[3] Weyl spinors being irreducible representations of $SL(2,\mathbb{C})$

spacetime.[4] Both the TB and the T*B at a point are both real vector spaces, $V$ and $U$ of the same dimension and therefore isomorphic to each other via many possible isomorphisms. That space-time, M itself affords a Geometric Algebra can be summarised by considering the aggregate of its multi-vector structures,

[scalars, vectors, bivectors, trivectors, pseudoscalar-volume form]

on the linear (co)-tangent spaces U,V to space-time, M. Given U we can construct the exterior space $\Lambda U$, the space of aggregates of multivectors of these different orders (0 to 4) of U that is closed under the exterior $\wedge$ product. In this way $\Lambda U$ becomes an algebra, the Grassman or exterior algebra. The introduction of a Riemannian, $g$ or symplectic , $\epsilon$ metric gives rise to a natural isomorphism between the tangent space and the cotangent space at a point, associating to any tangent co-vector a canonical tangent vector. This is implicit[5] in the defining of an oriented $\mathbb{C}$-valued version of a 4-volume pseudo-scalar $\eta \in \Lambda^4 U$ of space-time through the use of the Hodge dual operation $^*$,

$$\eta = {}^*(\theta^a \wedge \theta^b \wedge \theta^c \wedge \theta^d) := \frac{1}{4!}\phi\tilde{\epsilon}_{abcd}\theta^a \wedge \theta^b \wedge \theta^c \wedge \theta^d$$
$$\phi\tilde{\epsilon}_{abcd} \leftrightarrow i(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'}).$$

Typically $\phi$ would be the square root of the modulus of the metric determinant, $\sqrt{g}$ a tensor density of weight +1 to offset the -1 weight of $\tilde{\epsilon}$. In standard metric-affine formulations the spacetime metric emerges from the a priori metric symplectic spinors according to $g_{ab} \leftrightarrow \epsilon_{AB}\epsilon_{A'B'}$.

In this spirit, the variational formulation of Plebanski [5] uses $^*$ as an indempotent algebraic structure splitting the local algebra of the complexified $SO(1,3)_{\mathbb{C}} \xrightarrow{\sim} SO(4,\mathbb{C})$ gauge bundle over space-time, M into left and right handed ideals,

$$so(1,3)_{\mathbb{C}} = so(1,3\mathbb{C})^+ \oplus so(1,3\mathbb{C})^-.$$

The fully chiral dynamical variable, a basis of anti-self dual two-forms $\Sigma^{AB}$, is to be interpreted as the gauge-potential field which in the weak field limit possesses an excited "graviton" state.[6]

The Geometric Algebra of physical 3-space, that is the Clifford algebra $Cl_3$ is sufficient to describe physical 4-space-time because as (1) shows for the universal covering group it forms a self-adjoint subspace, [40]: the space $Cl^*{}_3 := GL(2,\mathbb{C})$ of "rotors", an 8-dimensional real Lie group of which $SL(2,\mathbb{C})$ is a six parameter subgroup,

$$GL(2,\mathbb{C}) := \{\xi \in GL(2,\mathbb{C}) | \xi^A{}_B = fL^A{}_B,\ f^2 = det(\xi^A{}_B)\}.$$

$GL(2,\mathbb{C})$ transformations can thus be decomposed into $SL(2,\mathbb{C}) \otimes \mathbb{C}$ according to,

[4] for a configuration space of a generic field, $\phi^A$ the fibre bundle of frames is $\pi : B \to M$. While the cotangent bundle T*B of the symplectic geometry of phase space does not need a metric structure on the basic world sheet, M to define the differential of a function, a metric is needed in order to define the gradient on its (dual) tangent bundle, TB. As such the tangent co-vector of T*B is called the canonical one-form or symplectic potential and can be viewed as that primitive object from which the metric structure on the base is derived.

[5] $\leftrightarrow$ idicates isomorphism between Levi-Cevita tensor density and symplectic spinor metric.

[6] That is, once reality conditions are applied "off-shell" to an otherwise complex field.

$$\xi^{\tilde{A}}{}_B = f L^{\tilde{A}}{}_B \quad \text{and} \quad (\xi^{-1})_{\tilde{B}}{}^A = f^{-1}(L^{-1})_{\tilde{B}}{}^A.$$

The homomorphism of $Cl^*{}_3$ into the 7-dimensional Unimodular group, $UL(2,\mathbb{C})$ of Lorentz dilations, being the product, in any order, of an element of the restricted Lorentz group and a homethety. While $UL(2,\mathbb{C})$ is a subgroup such that $|\xi| = 1$ where weights and anti-weights overlap, the spinorial group, $SL(2,\mathbb{C})$ transforms with $\xi = 1$. The relevant machinery of spinor-density valued objects was developed in unpublished notes of Plebanski, [6] the salient features of which are collected in the article.

Noteworthy in the above constructions is that a "metricity" constraint fixing the dynamical connection as metric compatible $Q_{ab} = {}^{\Gamma}\nabla g_{ab} = 0$, so that Weyl Dilations find no physical representations needs to be put in by hand, external to a variational formalism. The constructions to be developed in the following are distinct from that of Weyl who previously tried to relate Maxwell to linear connections which were non-metric, [3]. Indeed this need not be the case as we could define the symplectic spinorial covariant derivative according to, ${}^{\Gamma}\tilde{\nabla}\epsilon_{AB} = Q_{AB} \neq 0$ but such a further variation will not be entertained here and which interestingly has not been considered in non-chiral developments by Smolin et al, [39].

## Machian and Wave Eigenstate convergence

Einstein's General Theory is built on the Equivalence Principle, *ab initio* locally defined it rests uneasily with its concomitant non-local radiation predicted features: a unit charge hanging on a thread attached to the ceiling of a freely falling Einstein lift, to an observer in the lift, does not appear to radiate as it is at rest but, viewed by an observer on the Earth it radiates falling down as it is with constant acceleration, g. It is Boundary Conditions, rather than the Field Equations that embody and identify a wave's nature.[7]In the case of gravitational waves, this (linearised) gravitational field is required to possess a quadrupole moment.[8]

Requiring that a gravitational wave should satisfy boundary conditions at infinity[9] means, in the presence of electromagnetic radiation, a spacetime has a far from source Ricci tensor in the form of a null dust with double null vector. The gravitational plane wave has a 5-dimensional group of isometries just as a plane electromagnetic wave has a 5-dimensional group of symmetries: Bondi et al inspected all Ricci flat metrics

[7]Motivated by this inconsistency, Brans-Dicke and Sciama's search for a theory closer to Einstein's Machian pretensions, a theory of inertia, was one in which local experiments could be interpreted as giving information about the universe as a whole.Other variants of Einstein's General Theory look to more fully encapsulate Mach's principle of inertia (following the work of Brans-Dicke), by offering extra degrees of freedom to far field solutions of Gravitational Waves, (GW) or by offering predictions disinguishable from GR in respecting stronger versions of Einstein's Equivalence Principle. The 2018 sourced gravitational and electromagnetic wave (gamma ray bursts) from a neutron-star merger detected by LIGO, [32] and the space-based Fermi satellite confirmed the speeds of GW and EW as equal comparable down to 15 orders of magnitude precision: just as an EW is an oscillation in the $\mathbf{E}$ and $\hat{\mathbf{B}}$, captured spinorially in the bi-vector Faraday field, $\phi_{AB}$ that propagates at the speed of light, $c = \frac{1}{\sqrt{\epsilon\mu}}$, so is a GW an oscillation in the gravitational field, Weyl Curvature spinor, $\Psi_{ABCD}$.

[8]Such that the GW, in its standard interpretation, when passing through any point in space, would both stretch space in one direction and compress the space in the orthogonal direction. Alternative theories of gravity can describe in addition to this "tensor" mode, vector and scalar polarised modes determining how the GW distorts spacetime and in what direction it can move in as it propagates.

[9]A generalisation of Sommerfelds radiation conditions

with symmetries of dimension greater than or equal to 4, (as given by Petrov[10].) and found exactly one class of solutions with the same 5-dimensional group isomorphic to the symmetry group of the electromagnetic eld. The class of metrics obeying this defintion of a plane gravitational wave depends on two free functions of one variable: the wave amplitude and its direction of polarization. The far from the source radiative (boundary) conditions selects out the algebraic special criterion, picking out from amongst the four special types those spacetimes designated as type N, with Weyl curvature tensor as the leading term at infinity. That both electromagnetic and gravitational radiation in certain spacetimes travel with the same speed of that of light in a vacuum has been described through an extension to Petrov classification schema.

## Handedness resurgence

The 2017/8 LIGO findings, at the very least do not repudiate the intertwinedness of electromagnetism and gravitation as two fields being describable in the Newtonian limit by a common Gaussian potential. As with the Standard Model, Dark Matter theories appear to bias a dilineation of gravitational over electromagnetic interactions even at weak field limits. To this end it is worth exploring the extent of a theory's reliance on its scalar-field and handedness content above and beyond any geometric naturalness considerations which have not borne fruit for the last forty years. The extra polarization states of GWs in the weak field limit motivate the construction of an intrinsically chiral scalar field in addition to normal GR tensorial-spin 2 modes.

## The ontology of a BF-like $GL(2,\mathbb{C})$ Gauge Theory

We are thus led to the construction of a Palatini style, Chiral Einstein-Maxwell Lagrangian comprising $GL(2,\mathbb{C})$ valued "metric-affine" potential with the charged scalar fields being most "naturally" described as complex-valued weighted-spinor densities. To summarise the framework to be developed, we have the decomposition of the $GL(2,\mathbb{C})$ connection,

$$\gamma^A{}_B = \Gamma^A{}_B + \delta^A{}_B \gamma^C{}_C,$$

that itself transforms as a covariant vector,

$$\gamma^{\tilde{A}}{}_{\tilde{B}} = \xi^{\tilde{A}}{}_A (\xi^{-1})_{\tilde{B}}{}^B \gamma^A{}_B + \xi^{\tilde{A}}{}_S d(\xi^{-1})_{\tilde{B}}{}^S,$$

in which the trace of the connection is identifed with a complex Faraday gauge potential $\gamma^C{}_C := \gamma = 2A$. The forms $\Gamma^A{}_B$ are defined up to $SL(2,\mathbb{C})$ transformations induced by $GL(2,\mathbb{C})$ according to the affine representation,

$$\tilde{\gamma} = \gamma - dln(\xi) \quad \text{and} \quad \Gamma^{\tilde{A}}{}_{\tilde{B}} = L^{\tilde{A}}{}_A (L^{-1})_{\tilde{B}}{}^B \Gamma^A{}_B(\epsilon) + L^{\tilde{A}}{}_S d(L^{-1})_{\tilde{B}}{}^S,$$

and the hermitian matrix of 1-forms, $\theta^{AB'} = \theta^{AB'}{}_a \theta^a$ is a weighted spinor density $(w = -\frac{1}{2},\ \bar{w} = -\frac{1}{2})$ under $GL(2,\mathbb{C})$,

[10]The Petrov classication consists in the enumeration of the distinct eigendirections of the Weyl tensor called principal null directions (PNDs). If at a point all four PNDs are distinct, the spacetime at this point is called algebraically general. If at least two of the PNDs coincide, the spacetime at this point is called algebraically special. Various coincidences of PNDs may occur, resulting in the stratication of the algebraically special spacetime points into four Petrov types with the Weyl tensor of a purely radiative spacetime being asymptotically of type N

$$\theta^{\tilde{A}\tilde{B}'} = \xi^{\tilde{A}}{}_{C}\theta^{CD'}\xi^{\tilde{B}'}{}_{D'} = fL^{\tilde{A}}{}_{C}\theta^{CD'}f'L^{\tilde{B}'}{}_{D'}.$$

In defining the determinant of the metric as per Urbantke [31] in terms of the basis bivectors of non degenerate $gl(2,\mathbb{C})$-valued $S$-forms, rather than the matrix of 1-forms, $\theta^{AB'}$ according to,

$$\sqrt{g} = \frac{i}{3}\{S^{A}{}_{B\alpha\beta}S^{B}{}_{A\gamma\delta} + \frac{1}{2}S_{\alpha\beta}S_{\gamma\delta}\}\epsilon^{\alpha\beta\gamma\delta}.$$

charged scalar fields, $\phi, \pi$ defined as complex valued weighted, $w = (1,0)$ -spinor densities (that may be interpreted as Higgs or Brans-Dicke dilaton in nature) may be simply added to give a fully $GL(2,\mathbb{C})$ chiral Einstein-Maxwell-Scalar Lagrangian,

$$f^{A}{}_{B} \wedge S^{B}{}_{A} + \frac{1}{2}\Xi_{B}{}^{A}{}_{D}{}^{C}S^{B}{}_{A} \wedge S^{D}{}_{C} + \sqrt{g}\{\bar{\pi}^{\mu}({}^{\gamma}D_{\mu}\phi) - \bar{\pi}^{\mu}\pi^{\nu}g_{\mu\nu} + \pi^{\mu}({}^{\gamma}D_{\mu}\bar{\phi})\}. \quad (2)$$

After outlining some of the motivation for the geometric approach being followed and introducing the chiral Lagrangian with its chiral coupling formulations to both fermionic and saclar fields, Gauge fixing and Integrability issues within this $GL(2,\mathbb{C})$ unified-field framework are outlined. The latter references the work of Lanczos, Bel-Robinson and Ellis in their respective construction of potentials of linearised gravitational field and Entropy super tensors. More techincal details associated to the G-structure of the theory resides in the appendices.

# Falling co-frame formulation of the General Theory

A field theory, be it classical or quantum, has as its final form a set of differential equations. If the dynamical theory is to autmatically possess conserved Noether currents the field equations will arise from a Variational principle. To hone its solution set to observed solutions supplementary constraints may be required. An action density involving first order derivatives of the gravitational metric potential is not a scalar density. It would be a scalar if formed straight from the contracted Riemann Curvature tensor, $\sqrt{-g}\mathcal{F}[g^{\mu\nu}]$ involving second order derivatives in $g_{\mu\nu}$. Einstein's introduction of the connection, $\Gamma^{a}{}_{b\mu}$ as a further independent dynamical variable comes at the loss of this invariance. Ellie Cartan's bundle formulation[11] in terms of elevator frame $\sigma^{a}{}_{\mu}$ read as,

$$\begin{aligned} {}^{\star}F^{a}{}_{b} \wedge \theta^{b} &= -8\pi T^{a}, \\ {}^{\Gamma}\nabla\eta^{ab} &= \eta^{abc} \wedge \Theta_{c} = -8\pi\tau^{ab}. \end{aligned} \quad (3)$$

The components of the energy-momentum, $T_{ab}$ and spin tensors, $\tau_{abc}$ are given in terms of the three forms $T_{a} = T_{ab}\eta^{b}$, $\tau_{ab} = \tau_{abc}\eta^{c}$. Here $\eta^{a} = {}^{*}\theta^{a}$ is the three form Hodge dual of the co-frame, $\theta^{a} = \sigma^{a}{}_{\mu}dx^{\mu}$, a differential form providing the local identification of space-time with local Lorentz quantities of the observer.

[11] For Lorenzian $a = (0,1,2,3)$, Lorentzian time-space indices of a freely falling frame and general co-ordinate indices $\mu = (0,1,2,3)$

# Cartan's Structure Equations

For Real General Relativity the soldering functor, $\sigma^a{}_\mu$ is required to be real ($\overline{\theta^a} = \theta^a$), $\overline{\sigma_\mu{}^{AA'}} = \sigma_\mu{}^{AA'}$. Imposing hermicity on $\theta^{AA'} = \sigma^{AA'}{}_\mu dx^\mu$ realises a real Lorentzian metric[12] given as $ds^2 = \epsilon_{AB}\epsilon_{A'B'}\theta^{AA'} \otimes \theta^{BB'}$. With $\Gamma^A{}_B$ and $\bar{\Gamma}^{A'}{}_{B'}$ (complex conjugate) $sl(2,\mathbb{C})$-valued connection one-form potentials with Torsion two form field, $\Theta^{AA'}$, the first Cartan structure equation reads

$$\Theta^{AA'} := d\theta^{AA'} - \theta_{AB'} \wedge \bar{\Gamma}^{A'}{}_{B'} - \theta_{BA'} \wedge \Gamma^A{}_B.$$

That is, $\Theta^{AA'} := \nabla\theta_{AA'}$, where $\nabla \equiv {}^\Gamma\nabla$ denotes the exterior covariant derivative with respect to the $sl(2,\mathbb{C})$-valued connection(s). The internal 'symplectic metric', $\epsilon_{AB}$ is given as fixed so that the internal $SL(2,\mathbb{C})$ connection is then traceless $\Gamma_{AB} = \Gamma_{BA}$ due to $\nabla\epsilon_{AB} = 0$. Defining the basis of anti-self dual two-forms as $\Sigma^{AB} := \frac{1}{2}\theta^A{}_{A'} \wedge \theta^{BA'}$, the second Cartan structure equations take the complex form,

$$\begin{aligned} \mathcal{F}^A_B &:= d\Gamma^A_B + \Gamma^A_C \wedge \Gamma^C_B, \\ \mathcal{F}^A_B &:= \Psi^A_{BCD}\Sigma^{CD} + \Phi^A_{BC'D'}\bar{\Sigma}^{C'D'} + 2\Lambda\Sigma^A_B + (\chi_D{}^A\Sigma_B{}^D + \chi_{DB}\Sigma^{AD}), \end{aligned} \tag{4}$$

where the curvature two-form, $\mathcal{F}^A_B$, has been decomposed into spinor fields of dimension 5,9,1 and 3 respectively, corresponding to the anti-self dual part of the Weyl conformal spinor, $\Psi^A_{BCD}$, the spinor representation of the trace-free part of the Ricci tensor, $-2\Phi^A_{BC'D'}$ and the Ricci scalar $24\Lambda$, - all with respect to the curvature of the $SL(2,\mathbb{C})$ connection and $\chi^{AB}$ arising from the presence of non-zero torsion. We note for now the 5 dimensions of $\Psi$, which of its five complex components $\Psi_0 ... \Psi_4$ defined across linearly-independent null tetrad fields, the $\Psi_4$ characterises an outgoing wave-like field. It turns out that $\Sigma^{AB}$, defined in terms of the co-frame dynamical variable, here for mere ease of exposition can be viewed as a dynamical basis variable of a variational principle whose interpretation is of a fully chiral dynamical "graviton" field object once reality conditions are applied "off-shell".

# Lagrangian of Chiral left-handed Dynamical Field

For a Maxwell Faraday field we can define a $\mathbb{C}$-valued anti-self dual Faraday field using a Hodge$^*$ structure,

$${}^-\mathbf{f}_{ab} = \mathbf{f}_{ab} + i^*\mathbf{f}_{ab}.$$

That the photon has helicity, s=1 is made apparent by expressing the F through the antisymmetric quantities $\epsilon_{AB}$ and $\epsilon_{A'B'}$, that provide symplectic structure[13] to two-dimensional spin space,

$${}^-\mathbf{f}_{ab} \leftrightarrow \phi_{AB}\epsilon_{A'B'}, \text{ and } {}^+\mathbf{f}_{ab} \leftrightarrow \tilde{\phi}_{A'B'}\epsilon_{AB}$$

where we write in abstract indices so a pair of capital spinor indices, unprimed and primed, stand for a single tensor index,

$$\mathbf{f}_{ab} \leftrightarrow \phi_{AB}\epsilon_{A'B'} + \tilde{\phi}_{AB}\epsilon_{AB}. \tag{5}$$

[12] Following [26] we consider this metric as nothing more than a paramterization of the gravitational field

[13] The isomorphism between Levi-Cevita tensor density and symplectic spinor metric is indicated by $\leftrightarrow$.

We can then see the decomposition of the gauge vector boson $(1,1) = (1,0) \oplus (0,1)$, the photon in this linearised theory, into left-handed (negative helicity, s=-1) and right-handed parts. To ensure the field $\mathbf{f}$ is a solution to the real field equations[14] we require that the $\mathbb{C}$-valued $\phi$ fields be complex conjugates of eachother ${}^*\mathbf{f}_{ab} = -i\mathbf{f}_{ab}$, that is $\phi^\star_{AB} = \tilde{\phi}_{A'B'}$. We have thus a symmetric spinor $\phi_{AB} = \phi_{(AB)}$ field, its two unprimed spin $\frac{1}{2}$ indices indicative of it being the anti-self dual part of the $\mathbf{f}$.

## Metric-Affine Gravity from a Topological BF field theory

In a gauge field theory of gravity, Trautman's Real Metric-Affine, Einstein-Cartan Lagrangian reads as [15],

$$\begin{aligned} \mathcal{L}_{EC} &= -\frac{1}{2}\mathcal{F}_{ab} \wedge \eta^{ab} = \mathcal{F}_{ab} \wedge \ *(\theta^a \wedge \theta^b), \\ &= L^+_{EC} + L^-_{EC} = -\frac{1}{2}[\mathcal{F}^+(\Gamma^+) + \mathcal{F}^-(\Gamma^-)] \wedge \eta^{ab}, \\ &= -\frac{i}{2}[\mathcal{F}^+_{ab} \wedge \Sigma^{ab} - \mathcal{F}^-_{ab} \wedge \Sigma^{ab}], \\ &\leftrightarrow -i[\mathcal{F}_{A'B'} \wedge \Sigma^{A'B'} - \mathcal{F}_{AB} \wedge \Sigma^{AB}]. \end{aligned} \tag{6}$$

Capovilla R, Dell J, Jacobson T, [8] Complex Lagrangian[16] follows,

$$\begin{aligned} \mathcal{L}_{SSJ} = 2\mathcal{L}_{EC} &= i\theta^A{}_{A'} \wedge \theta^{BA'} \wedge \mathcal{F}_{AB} \\ &= \mathcal{L}_{EC} - \frac{i}{2}d(\theta^a \wedge \Theta_a) + \frac{i}{2}\Theta^a \wedge \Theta_a, \end{aligned}$$

from which Plebanski's, [5]

$$\frac{i}{2}\mathcal{L}_\Sigma(\Sigma, \Psi, \Gamma) = \{\Sigma^{AB} \wedge \mathcal{F}_{AB} - \frac{1}{2}\Psi_{ABCD}\Sigma^{AB} \wedge \Sigma^{CD}\} \tag{7}$$

formulation in terms of the basic field variables $sl(2,\mathbb{C})$ valued bivector and connection can be inferred. The Lagrange Multiplier term, $-\frac{1}{2}\Psi_{ABCD}\Sigma^{AB} \wedge \Sigma^{CD}$ forces the Ricci part of the Curvature two-form to vanish and crucially the constraint arising from the variation (of what turns out to be the Weyl Curvature spinor of the gravitational field), $\Psi$ dictates that $\Sigma^{AB}$ is determined by a tetrad, up to $\overline{SL(2,\mathbb{C})}$ transformations on primed indices. The wholly Chiral nature of the Pebanski formulation is in the sense that local Lorentz representations involve only $SL(2,\mathbb{C})$ and not its conjugate, $\overline{SL(2,\mathbb{C})}$. That is, the dynamical field object rather than being a mixed index co-frame, $\theta^{AA'} = \sigma^{AA'}{}_\mu dx^\mu$, from which the (real Lorentzian) metric is derived "on-shell" as $ds^2 = \epsilon_{AB}\epsilon_{A'B'}\theta^{AA'} \otimes \theta^{BB'}$ is defined a priori as the anti-self dual two-form, $\Sigma^{AB}$; the archetypal B field in BF theory. This is discussed extensively in [9].

[14] Within the $U(1)$ gauge field theory of electromagnetism there is a need for supplemental "gauge fixing" conditions to hone the solution set to only those helicity states realised in nature. Without such "simplicity constraints" there will be unavailable polarisation states afforded to $\mathbf{f}$. Most of the quantitative physical predictions of a gauge theory can only be obtained under a coherent prescription by suppressing or ignoring these unphysical degrees of freedom.

[15] The $\mathbb{R}$-valued bivectors themselves are simple in the sense that they can be constructed from the verbein co-frame one forms $\theta^a = \theta^a_\mu dx^\mu \in \Lambda U$ as $\Sigma^{ab} = \theta^a \wedge \theta^b \in \Lambda^2 U$.

[16] Written merely conveniently in terms of the simple complex valued bi-vectors, $\Sigma^{AB}, \Sigma^{A'B'}$

## Non-Chiral or Off-Shell Chiral Reality

Diffeomorphism invariant topological field theories such as BF theory possess an absence of local degrees of freedom. In its rawest geometrised form the $\mathbb{R}$eal space-time manifold carries on it a $SL(2,\mathbb{C})$ spin (trivial vector) bundle, $B$ and its conjugate $\bar{B}$ associated to this $PB$. The tensor product of these two bundles is then identified with the complexified tangent bundle. Each fibre, $S \equiv \mathbb{C}^2$ of B consists of a 2-complex dimensional vector space equipped with the symplectic metric, $\epsilon_{AB}$. For illustration sake, if one reverses the view, observing the construction of the non-chiral Lagrangian built of complex form fields from a complexification of the real tetrad on the manifold, we see that from the real $\eta$[17] of (6) we can form the complex $\eta + i^*\eta \in \Lambda^4 U_{\mathbb{C}}$ which splits the complexified space of bivectors, $\Lambda^2 U_{\mathbb{C}} = \Lambda^2_- U_{\mathbb{C}} + \Lambda^2_+ U_{\mathbb{C}}$ into pairs of simple $\mathbb{C}$-valued bi-vectors,

$$^-\Sigma^{ab} = \frac{1}{2}(\Sigma^{ab} + i^*\Sigma^{ab}) = \frac{1}{2}(\Sigma^{ab} + i\eta^{ab}) \leftrightarrow \Sigma^{AB}\epsilon^{A'B'} \in \Lambda^2_- U_{\mathbb{C}}.$$

For Plebanski's complex BF-like Lagrangian, (7) to remin chiral as a Variational principle for Real General Relativity the following reality (non-chiral) conditions on the $SL(2,\mathbb{C})$ valued two forms, $\Sigma^{AB}$ need to be put in by hand, to ensure a real Lorentzian space-time,

$$\Sigma^{AB} \wedge \overline{\Sigma}^{A'B'} = 0, \text{ and } \Sigma^{AB} \wedge \Sigma_{AB} + \overline{\Sigma}^{A'B'} \wedge \overline{\Sigma}_{A'B'} = 0. \tag{8}$$

That is, they do not follow naturally as Euler-Lagrange equations and must be imposed "off-shell". It is these "simplicity" constraints (8) that enables a topological (BF) field theory to be a "BF-like" theories with local degrees of freedom. Smolin [38] and [39] et al have embraced these reality conditions within an $U(2)$ symmetry breaking non-chiral BF Lagrangian. Their non-chiral Plebanski action is thus more a theory of two metric "forms" with the presence of the simplicity constraints forcing them to coincide. Parity breaking arises at the level of the field equations and as such has much to commend in its naturalness.

## Warding off Gauge redundancy

The off-shell[18] imposition of reality conditions on $\Sigma$ is not a an irredeemably unique negative feature[19] of the Plebanski's chiral formulation. We note the prevelance of

[17] The defining of an oriented $\mathbb{C}$-version of the 4-volume pseudo-scalar $\eta \in \Lambda^4 U$ of space-time,

$$\eta = \frac{1}{4!}\phi\tilde{\epsilon}_{abcd}\theta^a \wedge \theta^b \wedge \theta^c \wedge \theta^d = \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3,$$

assumes this geometric isomorphism. In standard metric-affine formulations, $\phi$ is the square root of the modulus of the metric determinant, $\sqrt{g}$ while in purely affine connection formulations the density comprises the Ricci tensor.

[18] In quantum field theory, virtual particles are termed off shell because they do not satisfy the relativistic energy-momentum relation while real exchange particles do satisfy this relation and are termed on shell (mass shell).

[19] An example is the "more real" quadratic Dirac spinor Lagrangian, (QSL), a chiral real Hilbert-Palatini action when the metric is real being an on-shell quadratic function of spinor one-forms, without requiring that the space-time connection satisfy its equation of motion. QSL was originally constructed with a $sl(2,\mathbb{C})$-valued connection and was generalized to a $gl(2,\mathbb{C})$-valued connection, [35].

the "gauge fixing" procedure within classical field theories. Even after applying the Lorentz gauge fixing condition to a semi-classical $U(1)$ gauge theory to allow only observable transverse polarized Electromagnetic waves, still unphysical states of " longitudinal", and "time-like" modes in the $\mathbf{E}$ and $\hat{\mathbf{B}}$ field strengths need to be suppressed by auxiliary constraints known as Ward identities. These constraints act to reduce the traceable phase space of the realised fields. Indeed comparable constraints to these resolved a key obstruction to the twistor description of gravitational waves: called the googly problem, the requirement that a twistor description of right-handed interacting massless fields (with positive, right-helicity), uses the same twistor conventions that give rise to left-handed fields (negative helicity) was achieved by appending standard ad-hoc Ward constructions within Penrose's Palatial Twistor program, [25].

# Coupled Graviton-Photon Constrained BF theory

Viewed as a constrained BF theory, Plebanski's formulation lends itself to natural generalisations as in the Einstein-Maxwell theory of Robinson, [14]that employs the gauge group of $gl(2,\mathbb{C}) = sl(2,\mathbb{C}) \oplus \mathbb{C}$ field valued variables, with the $GL(2,\mathbb{C})$-valued $S^A{}_B$-forms being the primary field variable determined up to $\overline{GL(2,\mathbb{C})}$ transformations on primed indices. With a $GL(2,\mathbb{C})$-valued connection, $\gamma^A{}_B$ the chiral Lagrangian for Einstein-Maxwell reads

$$\frac{i}{2}\mathcal{L}_S(S,\alpha,\gamma) = f^A{}_B \wedge S^B{}_A + \frac{1}{2}\Xi_B{}^A{}_D{}^C S^B{}_A \wedge S^D{}_C. \tag{9}$$

The scalar Lagrange multiplier field term, $\Xi$ is needed ensures the basic two form field variable is derived from a co-frame. Such a chiral formulation leads to complex vacuum field equations for a complex metric with the real theory recovered only upon imposing (by hand) reality conditions[20] . As just discussed to ensure that the metric is both real and Lorentzian the following "simplicity" conditions are required,

$$S^{AB} \wedge \overline{S}^{A'B'} = 0, \text{ and } S^{AB} \wedge S_{AB} + \overline{S}^{A'B'} \wedge \overline{S}_{A'B'} = 0.$$

Again, if Chirality of Lagrangian is to be preserved, such "Ward-Identity like" reality conditions on the $GL(2,\mathbb{C})$ valued two forms, $S^{AB}$ remain off shell.

## Coupling of Chiral Charged Fermions

The Lagrangian for fermionic matter has a dependence on the connection so admits Torsion contributions but nevertheless can be written as the sum of a semi-chiral complex Lagrangian for vacuum General Relativity, $L_{SC}(\theta,\Gamma)$, a complex (semi)chiral fermionic matter Lagrangian, $L_{\frac{1}{2}}$ and a term, $L_{J^2}$ that ensures the standard Einstein-Weyl form of the field equations. Work such as this on $SL(2,\mathbb{C})$ BF two style chiral variational principles was extended to include various matter fields by Capovilla et al and Pillin, [4] and is in the appendices for reference. Following through with their general prescriptions we can effect the coupling of fermionic matter fields within an extended $GL(2,\mathbb{C})$ BF formulation. To do so we will need to formalise the idea of

[20] Again, the point of view to be taken here is that the symplectic metric $\tilde{\epsilon}_{AB}$ of the spin bundle is fixed once and for all and that the soldering form, (soldering form) $\tilde{\sigma}^{AA'}{}_\mu$ contains all the information pertaining to the metric $g_{\mu\nu}$ on space-time, $M$ itself. See alternatively Penrose [23].

a weighted spinor density and present the transformation properties of a $GL(2,\mathbb{C})$ connection for the bi-vector "graviton-photon" $S$-field. We gather together the spinor density structures arising from a generlaised spinor's transformation properties in the appendices. We highlight that it will be the case that the coupling of the "graviton-photon" to charged fermions naturally necessitates the restriction of the gauge group, $GL(2,\mathbb{C})$ to $SL(2,\mathbb{C})\otimes U(1)$ in order that final theory recovered is a real one. The $GL(2,\mathbb{C})$-valued two-form chiral Lagrangian for Einstein-Maxwell with chiral spinor field source is,

$$\begin{aligned}\mathcal{L}_{S\rho}(S,\gamma,\rho,\tau,\alpha) = -2i\mathcal{L}_S + (S^B{}_A + \frac{1}{2}\delta^B{}_A S^E{}_E)\wedge\rho^A\wedge\ {}^\gamma\mathcal{D}\lambda_B \\ + \tau_A{}^B{}_C\wedge S^A{}_B\wedge\rho^C + \frac{3}{32}\lambda_C\lambda^C\rho_A\wedge\rho^B\wedge S^A{}_B.\end{aligned} \tag{10}$$

Here ${}^\gamma\mathcal{D}$ represents the exterior covariant derivative associated to the $gl(2,\mathbb{C})$-value connection $\gamma^A{}_B$. The transformation properties Connection Potential Gauge are collected int he appendices. The spin $\frac{1}{2}$ fields are Grassman-odd objects and the chiral spin $\frac{1}{2}$ field quartic term has been included in order that the Einstein-Maxwell-Weyl field equations can be obtained from a first order Lagrangian. The field equation arising from the variation of one form $\tau_{ABC} = \tau_{(ABC)}$ means that the right-handed fermion may be represented as a left handed one form according to $\rho^C = \theta^{CC'}\tilde{\lambda}_{C'}$. These issued are discussed extensively in [7] and a related discussion for spin $\frac{3}{2}$ fields is included in the discussion notes. Now since we require a real theory, ${}^\gamma\mathcal{D}$ is extended to ${}^\Gamma\nabla$ as well as to act on both primed and unprimed spinors. The field equation resulting from the variation of $\gamma$ is

$${}^\gamma\mathcal{D}\eta^{AA'} = {}^\Gamma\nabla\eta^{AA'} - \eta^{AA'}(A+\bar{A}), \tag{11}$$

where

$${}^\Gamma\nabla\eta^{AA'} \equiv \frac{3}{4}J^{AA'} \quad \text{and} \quad K_{ABCC'} = \frac{1}{4}\epsilon_{C(A}J_{B)C'}. \tag{12}$$

It turns out that the correct charged Weyl equations written in terms of the connection $\gamma^A{}_B := \omega^A{}_B + \delta^A{}_B A$ (and its complex conjugate) are obtained from the variations of $\lambda$ and $\bar{\lambda}$,

$$\begin{aligned}{}^\gamma\mathcal{D}^{BA'}\bar{\lambda}_{A'} = 0 \quad \text{and} \\ {}^\gamma\mathcal{D}^C{}_{D'}\lambda_C = 0,\end{aligned}$$

only when the gauge group is restricted to $SL(2,\mathbb{C})\otimes U(1)$ so that the the latter term in (11) vanishes. We have then the (real) linear connection, associated to ${}^\gamma\mathcal{D}$ is then the $SO(1,3)$ connection, $\Gamma^a{}_b$.

## Scalar field coupling

With charged scalar fields, $\phi,\pi$ defined as complex valued spinor densities a chiral Einstein-Maxwell-Scalar Lagrangian for (charged) scalar fields reads,

$$\mathcal{L}_S(S,\alpha,\gamma) + 2i\mathcal{L}_H,$$

$$\mathcal{L}_H = \sqrt{g}\{\bar{\pi}^\mu({}^\gamma D_\mu\phi) - \bar{\pi}^\mu\pi^\nu g_{\mu\nu} + \pi^\mu({}^\gamma D_\mu\bar{\phi})\}, \tag{13}$$

for $gl(2,\mathbb{C})$-valued $S$-forms the configuration dynamical variables of the theory. Such scalar fields, may be considered as charged Higgs or of Brans-Dicke dilaton type being defined as spinor density-valued 0-forms, with $\phi$ having weight $w = (1,0)$ and its complex conjugate, $\bar{\phi}$ having $w = (0,1)$. We have an action of an exterior covariant derivative on a complex scalar field, $\phi$ with no spinorial indices upon which to contract. In order to make explicit the primacy of the two form as the dynamical field to be varied, $S^A{}_B = \Sigma^A{}_B + \epsilon^A{}_B\Sigma$ in $\mathcal{L}_H$ with the metric tensor $g_{\alpha\beta}$ derived as an 'on-shell' solution to the Euler-Lagrange equations, we require at the outset expressions for the metric density and its determinant soley in terms of these two forms. Their expressions are analogous to those produced by Urbantke [31] for chiral $sl(2,\mathbb{C})$-valued $\Sigma$-self-dual bivector forms,

$$\sqrt{g}g_{\alpha\beta} = \frac{2i}{3}\epsilon^{\mu\rho\sigma\nu}\{S^A{}_{B\alpha\mu}S^B{}_{C\rho\sigma}S^C{}_{A\beta\nu} - \frac{1}{4}S_{\alpha\mu}S_{\rho\sigma}S_{\beta\nu} -$$

$$\frac{1}{2}(S^A{}_{B\alpha\mu}S^B{}_{A\rho\sigma}S_{\beta\nu} + S^A{}_{B\alpha\mu}S_{\rho\sigma}S^B{}_{A\beta\nu} + S_{\alpha\mu}S^A{}_{C\rho\sigma}S^C{}_{A\beta\nu})\},$$

$$\sqrt{g} = \frac{i}{3}\{S^A{}_{B\alpha\beta}S^B{}_{A\gamma\delta} + \frac{1}{2}S_{\alpha\beta}S_{\gamma\delta}\}\epsilon^{\alpha\beta\gamma\delta}.$$

With an $\underline{\text{arbitrary}}$ selected but preferred handedness chosen (self-dual forms, determined up to $\overline{GL(2,\mathbb{C})}$ transformations on primed indices) we see that we are constructing the complex spinor-valued form equivalent of an oriented pseudo-tensor density. With the determinant of the metric defined in terms of our fundamental bivectors the (charged) scalar fields are complex valued weighted-spinor densities and may be interpreted as Higgs or Brans-Dicke dilaton in nature.

# Discussion

## Integrability Hierarchy of Graviton-Photon plane waves

Both Maxwell and Einstein's field equations are first order partial differential equations in respective field strengths, $\mathbf{f}$ and Riemmanian Curvature $\Omega^A{}_B = \frac{1}{2}R^A{}_{B\mu\nu}dx^\mu \wedge dx^\nu$ 2-forms comprised of gradients of potentials:

$$\mathbf{f}[\partial A] \leftrightarrow \Gamma[\partial g],$$

that are subject to Bianchi Identity Integrability conditions,

$$d^*\mathbf{f} = 0 \leftrightarrow^\omega \nabla\Theta^{AA'} = \Omega^A{}_B \wedge \theta^{BA'} + \Omega^{A'}{}_{B'} \wedge \theta^{AB'} = 0.$$

With these satisfied, the position-space picture of the wave function of a massless particle with helicity 2s, expressible in the 2-spinor form is

$${}^\Gamma\nabla^{AA'}\psi_{AB..F} = 0,$$

where we have total symmetry for each of the $|2s|$-index quantities,

$$\psi_{AB..F} = \psi_{(AB..F)}.$$

As was the case for the "photon" $|s|=1$, (see 5) for helicity spin, $|s|=2$ we have,

$$C_{abcd} \leftrightarrow \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \tilde{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD},$$

delivers us the weak-field (linearized) Einstein vacuum equations which are read as ($|s|=1$ or $|s|=2$ equations) wavefunctions having a positive-frequency conditions applied to them.[21] With the decomposition (30) the exterior covariant derivative for connection $\gamma$ for an arbitrary spinor density, $\psi$ endowed with complex weights $(w,\bar{w})$ reads,

$$^{\gamma}\mathcal{D}_{\mu}\psi^{A_1..A_kB_1'..B_l'}_{C_1..C_pD_1'..D_q'} = \{^{\Gamma}\nabla_{\mu} + [w + \frac{1}{2}(k-p)]A_{\mu} + [\bar{w} + \frac{1}{2}(l-q)]\bar{A}_{\mu}\}\psi^{A_1..A_kB_1'...B_l'}_{C_1..C_pD_1'..D_q'} \quad (14)$$

so that,

$$^{\gamma}\mathcal{D}^{AA'}\Xi_{ABCD} = 0 \quad (15)$$

is our positive helicity linearised photon-graviton wavefunction.

In GR we think of our real fourth rank Weyl tensor, $C_{abcd}$ or indeed it's (anti)-self dual $C^{\pm}$ version, evaluated at some event, as acting on the space of bivectors, $^{\pm}\Sigma^{ab}$ at that event, like a linear operator acting on a vector space. The problem is then to find eigenvalues $\lambda$ and eigenvectors ('eigenbivectors') $\Sigma^{ab}$ such that

$$\frac{1}{2}C^{ab}{}_{cd}\Sigma^{cd} = \lambda\Sigma^{ab}.$$

Multiplicities among the eigen-bivectors indicate an algebraic symmetry at the given event and are determined by solving the resulting characteristic (quartic) equation. The eigen-vectors are the principal null directions that are classified according to six possible Petrov types characterised by the algebraic symmetry according to their (anti)-self dual Weyl parts. With the $GL(2,\mathbb{C})$-valued $S$-form the characteristic equation reads as,

$$\frac{1}{2}\Xi^{AB}{}_{CD}S^{CD} = \lambda S^{AB} \quad \text{for}$$

$$\begin{aligned} \Xi_{ABCD} &= \Psi_{ABCD} + \epsilon_{AB}\phi_{CD} + \phi_{AB}\epsilon_{CD}, \\ S^{AB} &= \Sigma^{AB} + \epsilon^{AB}\Sigma \end{aligned}$$

Two Polarization modes[22] are carried by the Weyl tensor of GR, in which spacetime distortion is in the plane perpendicular to the direction in which the gravitational wave

[21] The $\Psi$-part would then describe the left-handed (s=-2) component of the particle's wave function while $\tilde{\Psi}$ the s=2 component. As we did for $\mathbf{f}_{ab}$ we insist on $C_{abcd}$ being real so that $\tilde{\Psi}_{A'B'C'D'} = \Psi^{\star}_{A\bar{B}\bar{C}D}$, that is schematically $\overline{sl(2,\mathbb{C})} \leftrightarrow \epsilon^2 \leftrightarrow sl(2,\mathbb{C})$ through the application of the Lie dual twice to obtain the classical $\mathbb{R}$ solutions of the field equations. The part of $C_{abcd}$ involving $C^- = \Psi$ is called the anti-self-dual part. Being a spin-2 particle, means the graviton need only spin half a revolution in its spin (symplectic) space to mark out its locus of indistinguishability.

[22] One new report, LIGO publications tests for more than the two "tensor" polarisation modes that can be carried by $\Psi_4$. The geometry associated with plane-fronted transverse gravitational waves of the sort that LIGO are looking at are all Petrov type N solutions with vanishing Newman-Penrose scalar $\rho_{NP} = -n_{a;b}m^a\bar{m}$. Indeed the Weyl scalar $\Psi_4$ describing the outgoing gravitational radiation (in an asymptotically flat spacetime) reads as

$$\Psi_4 = \frac{1}{2}\left(\ddot{h}_{\hat{\theta}\hat{\theta}} - \ddot{h}_{\hat{\phi}\hat{\phi}}\right) + i\ddot{h}_{\hat{\theta}\hat{\phi}} =: -\ddot{h}_+ + i\ddot{h}_{\times} \ ,$$

in which the $h_+$ and $h_{\times}$ are the "plus" and "cross" polarizations of gravitational radiation and double dots represent double time-differentiation. These types of waves are closely analogous to our understanding of electromagnetic waves: the gravitational wave fronts move along the congruence of null curves in an irrotational (twist)-free way.

travels. The "vector" $(1,0)$ modes could be accomodated in the spinor connection $GL(2,\mathbb{C})$-valued 1-form,

$$\gamma^a{}_b \leftrightarrow \gamma^a{}_{b\mu}dx^\mu = \gamma^A{}_B\epsilon^{A'}{}_{B'} + \gamma^{A'}{}_{B'}\epsilon^A{}_B,$$

with the spinor term $\chi^{AB}$ from Cartan's second structural equation, 4 and in 33 arising from the presence of non-zero torsion for the sourceless far-field solutions pertinent to gravitational waves in which the Ricci tensor is zero $\Phi = 0$. The deployment of additional constraints akin to those of the Ward Identities for Yang-Mills theories will be required for our wave bivector with principle null direction, $\alpha_B$

$$\Xi_{ABCD}S^{CD} = (\alpha_A\alpha_B\alpha_C\alpha_D S^{CD}{}_{\mu\nu} + \chi_{D(A}S_{B)}{}^D{}_{\mu\nu})dx^\mu \wedge dx^\nu.$$

While from 34 we see there are 10 independent components of Weyl curvature encoded in 5 complex components, $\Psi$ and six independent components of the Faraday-Maxwell 2-form (electromagnetic field strength tensor) $\mathbf{f}_{ab}$. The latter encoded in three complex Maxwell-Newman Penrose scalars, $\phi_0 = \mathbf{f}_{ab}l^a m^b$, $\phi_1 = \frac{1}{2}\mathbf{f}_{ab}(l^a n^b + \bar{m}^a m^b)$, $\phi_2 = \mathbf{f}_{ab}\bar{m}^a n^b$. We note the pnds $\alpha_B$ as such represent both propogating Petrov type N space-time with EM waves.[23] Indeed this is as it should be: the gravitational plane wave has a 5-dimensional group of isometries just as the plane electromagnetic wave has a 5-dimensional group of symmetries. Of all the Ricci flat metrics with symmetries of dimension greater than or equal to 4, (as given by Petrov) there is only one such class of solutions with the same 5-dimensional group isomorphic to the symmetry group of the electromagnetic field.

## Lanczos Potential for linearised Gravitational Field

In the context of a Metric-"Affine" field theory of gravitation we have seen that $C^\pm$ "carries" the gravitational wave. Issues with gauge fixing and finding a source potential for the (far-field linearised) gravitational wave (solution) have been treated by Lanczos through his construction of a 3-tensor $L_{abc}$, the potential for the Weyl conformal tensor $C_{abcd}$. We summarise the spinor treatment reviewed in Edgar, A. Hoglund (see references therein). The argument is that any candidate symmetric 4-spinor $X_{ABCD}$ admits locally a solution to

$$X_{ABCD} = 2^\Gamma\nabla_{(A}{}^{A'}L_{BCD)A'}, \tag{16}$$

where $L_{BCDA'}$ needs to be symmetric in all unprimed indices according to the satisfaction of the "Lanczos algebraic gauge",

$$L_{ab}{}^b = 0, \tag{Algebraic-Gauge}$$

The brackets in 16 can be omitted by further fixing a "Lanczos differential" gauge

$$L_{ab}{}^c{}_{;c} = 0 \leftrightarrow^\Gamma \nabla^{AA'}L_{ACDA'} = 0. \tag{Differential-gauge}$$

[23] For examples of approaches that capture electromagnetic and gravitational field vibrations in a Clifford-spinor setting see quadratic spinor for Einstein-Maxwell. For a unified gauge potential approach to the Einstein Maxwell equations and for an approach that uses the Dirac spinor representation of the the Clifford algebra in a space-time admitting Torsion see Rocha.

We can require further that $X_{ABCD}$ satisfies a Bianchi-type equation with source current, $I$ of the form,

$$^{\Gamma}\nabla_{AA'}X^{A}{}_{BCD} = J_{BCDA'}.$$

The $L_{BCDA'}$, in Lanczos algebraic gauge but arbitrary differential gauge, then satisfies the wave equation,

$$\Box L_{BCDA'} - 3^{\Gamma}\nabla_{(B|A'|}\,{}^{\Gamma}\nabla^{EZ'}L_{CD)EZ'} - 6\Phi_{AF'A(B}L_{CD)}{}^{AF'} + \frac{1}{4}RL_{BCDA'} = I_{BCDA'}, \qquad \text{(Algebraic-Wave)}$$

In vacuum, so that $\Phi_{ABA'B'} = 0 = R$ and with both Lanczos gauges chosen, the wave equation[24] takes the very simple D'Almbertian form

$$\Box L_{BCDA'} = J_{BCDA'}. \qquad \text{(Alg+Differential-Wave)}$$

For the Maxwell-Faraday potential we have a field, $\mathbf{f}$ satisfying $d\mathbf{f} = 0$, $d^{*}\mathbf{f} = j$ as a Bianchi identity for $\mathbf{f} = dA$ of a gauge potential $A'_{\mu} = A_{\mu} + \partial_{\mu}\Lambda$. That is, a linear field equation satisfying,

$$^{\Gamma}\nabla^{AA'}\phi_{AB} = 0.$$

While its field integrability hierarchy is gauge independent. Only with gauge fixing is the Lanzcos potential Integrability hierarchy comparable for GR. For GR we have the trace-free Weyl tensor $C_{abcd} = C_{[cd][ab]}, C_{[abc]d} = 0, C_{abca} = 0$, satisfying the vacuum Bianchi identities: $^{\Gamma}\nabla_{[a}C_{bc]de} = 0$ delivering us a particular weak-field (linearized) Einstein vacuum wave form solution,

$$\Psi_4 := C_{\alpha\beta\gamma\delta}n^{\alpha}\bar{m}^{\beta}n^{\gamma}\bar{m}^{\delta}.$$

Given the symmetries of the extended curvature spinor both the Lanzos algebraic and differential gauges need to be fixed

$$\Xi_{ABCD}(= \Psi_{ABCD} + \epsilon_{AB}\phi_{CD} + \phi_{AB}\epsilon_{CD}) = 2^{\gamma}\mathcal{D}_{A}{}^{A'}L_{BCDA'},$$

for a Lanzcos "potential" for gravitional-electromagnetic waves to satisfy (Alg+Differential-Wave). And only if integrability of the generalised Weyl spinor, 34 Bianchi-type equation is applied,

$$^{\gamma}\mathcal{D}_{AA'}\alpha^{A}\alpha_{B}\alpha_{C}\alpha_{D} = \tilde{I}_{BCDA'}.$$

The necessity for both algebraic and differential gauge fixing muddies the integrability hierarchy picture just for linearised wave solutions and begs the question what are the meanings of these $L$- "potentials" with respect to the $S$-bivector forms?

## Bel-Robinson Entropy for combined gravitational-em fields

"Maxwell's"-stress energy tensor of pure electro-magnetic radiation, [37] is

$$T^{em}_{ab} = \frac{1}{4}(\mathbf{f}_{ae}\mathbf{f}^{e}{}_{b} +^{*} \mathbf{f}_{ae}{}^{*}\mathbf{f}^{e}{}_{b}),$$

and is formally similar to the Bel-Robinson super-energy tensor, its gravitational equivalent written as,

$$T^{g}_{abcd} = \frac{1}{4}(C_{eabf}C^{e}{}_{cd}{}^{f} + \ {}^{*}C_{eabf}\ C^{\star e}{}_{cd}{}^{f}), \qquad (17)$$

[24] ($\Box =^{\Gamma} \nabla^{\Gamma}_{a}\nabla^{a}$) is the d'Alembertian differential wave operator in four-dimensional spacetime.

where $C^{\star}_{abcd} = \frac{1}{2}\eta_{abef}C^{ef}{}_{cd}$ is the Lie Hodge-dual of the Weyl tensor and ${}^{*}C_{abcd} = \frac{1}{2}\eta_{efcd}C_{ab}{}^{ef}$ is its Hodge-dual[25]. As such, it is symmetric, tracefree, and covariantly conserved in the vacuum. We will consider the spinor equivalent of a generalised Weyl-Maxwell curvature super gravitation-em field in the discussion. In two components spinors the Bel-Robinson tensor, (17), simply reads,

$$T_{abcd} \leftrightarrow \Psi_{ABCD}\tilde{\Psi}_{A'B'C'D'},$$

and has been proposed as the covariant (if observer-dependent) object to describe the Entropy of the gravitational field in a [Gravitational Entropy Proposal]. $T^{a}{}_{bcd}$ and its divergence, $\nabla_{a}T^{a}{}_{bcd}$ arise soley as a consequence of the Integrability conditions of Einstein's equations, that is the Bianchi identities and their contractions. The components of the Bel-Robinson tensor, $T^{a}{}_{bcd}$ have dimension $l^{-4}$ so that $\frac{1}{\beta^2}T^{a}{}_{bcd}$ for $\beta = \frac{8\pi G}{c^4}$ has dimensions of the square of energy-momentum. As such when Einstein's equations are satisfied this motivates taking its (trace-free) 'square-root'. For the gravitational wave[26] case, $\Psi_4 = C_{abcd}\bar{m}^{a}l^{b}\bar{m}^{c}l^{d}$ this results in,

$$t_{ab} = |\Psi_4|n_a n_b,$$

Within a non-chiral $GL(2,\mathbb{C})$ spin-density formalism akin to the program of Smolin et al, [38] we note the combined entity

$$\tilde{T}_{abcd} \leftrightarrow \Xi_{ABCD}\tilde{\Xi}_{A'B'C'D'},$$

offers itself naturally as the entropy of the combined propogating $\gamma$ connection fields.

## Bundle Geometry Hierarchy

A Garrett Lisi, [1] argues for a modified BF theory action built ofa collection $E_8$-valued 2-forms and anti-Grassmann 3-form Lagrange multiplier fields, defined over a four dimensional base manifold. There is a natural joining of unified gravitational spin connection and the frame-Higgs while the magnitude of the Higgs, $\sqrt{(}\phi^2)$, is a conformal factor that can be absorbed into the magnitude of the frame. Lisi proposes a form of the action based on an approach to writing general relativity as a gauge theory, invented by MacDowell and Mansouri in which the metric and local Lorentz connection are unified in a larger deSitter connection, but the action is only invariant under a local Lorentz subgroup. This has echoes of the foregoing but contact with Lisi work is made harder with his extensive use of the Clifford product.[27]

[25] While the Equivalence Principle effectively intertwines the Hodge structure on the space-time (indexed by $\mu$) algebra,

$${}^{-}\mathcal{F} = {}^{-}\,\mathcal{F}_{ab}\theta^{a}\wedge\theta^{b} = (\mathcal{F}_{ab} + i^{*}\mathcal{F}_{ab})\sigma^{a}_{\mu}\sigma^{b}_{\nu}dx^{\mu}\wedge dx^{\nu} = \mathcal{F}_{\mu\nu}\Sigma^{\mu\nu} + i\mathcal{F}_{\mu\nu}\eta^{\mu\nu}$$

with the Hodge-Lie $\star$ structure on freely the falling (Lorentzian, indexed by $a$) gauge algebra,

$$\mathcal{F}^{-} = (\mathcal{F}_{ab} + i\mathcal{F}^{\star}_{ab})\sigma^{a}_{\mu}\sigma^{b}_{\mu}dx^{\mu}\wedge dx^{\nu}$$

we should be careful to distinguish the two.

[26] $\Psi_4$ is the only non-zero Weyl (complex) scalar for $n_a$ chosen to be aligned with the principal null directions along which a plane fronted wave propogates.

[27] By admitting a metric, such a product can be formed between multivector aggregates $\alpha$ and $\beta$ (such as between $\theta$ and $\Sigma$, say) within the one algebra, $\Lambda U$. For example, in the notation of the

## Postscript: Susceptibility of Hodge Duality

In our present material universe, of the four Maxwells equations only Faradays and modified Amperes laws are independent. Free charges and currents source the four electric and magnetic (resp.) fields and fluxes $E, D$ and $H, B$ which are neatly gathered as tensor-valued two forms, $F$ and $G$. The constraining system of equations that describe the behaviour of matter under the influence of these fields, known as the Constitutive (having the power to establish or give organised existence to something) relations. In the presence of external electric (magnetic) fields any permeated substrate susceptible medium becomes polarized (magnetized) such that the electric flux density of the medium is characterised by a polarisation vector indicating the dipole moment per unit volume as the displacement current, $D = \epsilon_0 E + P$. Similarly the magnetic flux density in a magnetic medium is $B = \mu_0 H + M$ for magnetisation vector, M captures the magnetic dipole moment per unit volume. In the absence of any material (i.e. in a vacuum) the relations are given by $D = \epsilon_0 E, B = \mu_0 H$ and whose invariant form is written using the Hodge dual, $G = \chi(^\star)F =^\star F$ the final equality being true because the vacuum susceptibility is assumed to be trivial. The end game eon is thus characterised by a phase change in which the universe again once again exhibits conformal symmetry. The polarisable spatial dielectric of our present epoch is bookended between conformal spaces of trivial susceptibility with Electric and Magnetic fields and fluxes wholly intertwined by the conformally invariant Hodge dual operator. The forgoing perhaps invites the following questions:

1) Is the gauge freedom that we can use to construct our Field strengths from an equivalence class of indistinguishable potentials only apparent Now, within this present material-immaterial epoch? That such Gauge ambiguity within classical field theories is nothing more than a misguiding mathematical artefact of a purer eon.

2) What are the timely (and timeless) quantities of ontological interest in a semi-classical theory of gravito-electrodynamics?

3) Just prior to the arrival of total radiation dominance of the universe at its conformal boundary, consider the universe of two material charged fermions and the last lone black hole. Consider fermions, as the only particle entities (in a non quantum theory) from which all fields arise. Ask then to what potential or field curvature object would we most cogently attribute the Ahranhov -Bohm effect on such a curved space-time?

# Appendices

## Appendix 1: Spinor-density structures

The transition functions of the $GL(2, \mathbb{C})$ bundle over M are represented by complex non-singular $2 \times 2$, matrices $(\xi^{\tilde{A}}{}_B)$ and contain 8 real parameters. The inverse and conjugates are denoted by, $(\xi^{-1})_{\tilde{B}}{}^A$, $\xi^{\tilde{A}'}{}_{B'}$ and $(\xi^{-1})_{\tilde{B}'}{}^{A'}$, their determinants are given

Clifford system we can write Maxwell's equations as a single quaternionic equation

$$\left(\partial_t + \nabla\right) \cdot \mathcal{F} = \frac{\rho}{\epsilon} - \mu c J.$$

in terms of a real Clifford[28] algebra-valued field, $\mathcal{F} = \mathbf{E} + \mathbf{jc}\hat{\mathbf{B}}$ for a wave that propogates at speed, $c = \frac{1}{\sqrt{\epsilon\mu}}$.

by

$$f^2 := \xi = det(\xi^A{}_B), \bar{f}^2 = det(\xi^{A'}{}_{B'}).$$

A generic spinor density transforms from (say) a null ($A$) to orthonormal ($\tilde{A}$) frame under $GL(2,\mathbb{C})$ according to,

$$\psi^{\tilde{A}\tilde{B}'}{}_{\tilde{C}\tilde{D}'} = (f)^{2w}(f')^{2\bar{w}}(\xi^{-1})_{\tilde{C}}{}^C \xi^{\tilde{A}'}{}_A (\xi^{-1})_{\tilde{D}'}{}^{D'} \xi^{\tilde{B}'}{}_{B'} \psi^{AB'}{}_{CD'}. \tag{18}$$

The independent complex numbers $(w, \overline{w})$ are the weight and anti-weight that characterise a generic field, $\psi^{AB'}{}_{CD'}$ whose complex conjugate is $\psi^{A'B}{}_{C'D}$ and carries weights $(\bar{w}', w')$. The (familiar $SL(2,\mathbb{C})$-valued spinors are objects that transform with vanishing weights so that $\delta^A{}_B$ and $\delta^{A'}{}_{B'}$ are numerical spinors with the same numerical value in all spinorial frames. Symplectic metrics are numerical spinor densities with $\epsilon_{AB}$ having weight $w = +1$ transforming according to,

$$\epsilon_{\tilde{A}\tilde{B}} = (\xi^{-1})_{\tilde{A}}{}^C (\xi^{-1})_{\tilde{B}}{}^D \epsilon_{CD} = (f^2)(L^{-1})_{\tilde{A}}{}^C (L^{-1})_{\tilde{B}}{}^D \epsilon_{CD}.$$

The other Levi-Cevita symbol densities have respective weights,

| | w | $\bar{w}$ |
|---|---|---|
| $\epsilon_{AB}$ | 1 | 0 |
| $\epsilon^{AB}$ | -1 | 0 |
| $\epsilon_{A'B'}$ | 0 | 1 |
| $\epsilon^{A'B'}$ | 0 | -1 |

and we are wary that raising (lowering) of an unprimed (primed) index increases (decreases) the weight (anti-weight) by one. A hermitian object is such that,

$$(\psi^{A_1 \ldots A_k B'_1 .. B'_k})^* = \psi^{A_1 \ldots A_k B'_1 .. B'_k},$$

holding in all spinorial frames only if the basic weights are related $w' = \bar{w} \leftrightarrow \bar{w}' = w$.

## Appendix 2: Metrical Gauge to Spin Group Morphisms

In order for $\theta^{AA'}$ to be treated as a spinor density under $GL(2,\mathbb{C})$ we must assign it weights such that $\bar{w} = w$. To see this note that the transformation of $\theta^{AA'}$ to a new spinorial frame means:

$$\begin{aligned}\theta^{\tilde{A}\tilde{A}'} = \xi^w \xi'^{w'} \xi^{\tilde{A}}{}_A \xi^{\tilde{A}'}{}_{A'} \theta^{AA'} &= \xi^w \xi'^{w'} \xi^{\tilde{A}}{}_A \xi^{\tilde{A}'}{}_{A'} \sigma^{AA'}_a \theta^a, \\ &= \xi^w \xi'^{w'} \xi^{\tilde{A}}{}_A \xi^{\tilde{A}'}{}_{A'} T^{-1a}_{\tilde{a}} \sigma^{AA'}_a \theta^{\tilde{a}}\end{aligned} \tag{19}$$

for $\theta^a = T^{-1a}_{\tilde{a}} \theta^{\tilde{a}}$ and $(T^{\tilde{a}}{}_a) \in SO^+(1,3)$ such that $det(T^{\tilde{a}}{}_{\tilde{b}}) = 1$ and $T^{\tilde{4}}{}_{\tilde{4}} \geq 1$ defining the proper orthochronous metrical ambiguity in the forms $\theta$ through $ds^2$. The homorphism[29] between $GL(2,\mathbb{C})[\infty^2 \leftrightarrow 1]SO^+(1,3)$ is effected by demanding that,

$$\begin{aligned}\sigma^{\tilde{A}\tilde{A}'} &= \sigma^{\tilde{A}\tilde{A}'}_a \theta^a, \text{ so that} \\ T^{\tilde{a}}{}_a &= -\frac{1}{2} \xi^w \xi'^{w'} \sigma^{\tilde{a}}{}_{\tilde{A}\tilde{A}'} \xi^{\tilde{A}}{}_A \xi^{\tilde{A}'}{}_{A'} \sigma^{AA'}_a,\end{aligned} \tag{20}$$

for $\sigma^{\tilde{A}\tilde{A}'}_a$ the standard Pauli matrices. This is the case if $(\xi)^{2w+1}(\xi')^{2w'+1} = 1$, which for arbitrary complex $\xi$ holds for $T^{\tilde{a}}{}_a$ if $w = -\frac{1}{2} = w'(= \bar{w})$ as advertised,

[29] A given $SO^+(1,3)$ may be induced by an $(\infty)^2$ of $GL(2,\mathbb{C})$ transformations, leaving two real parameters of $GL(2,\mathbb{C}$ undetermined and to be associated to electro-magnetic gauge potential freedom.

| | w | $\bar{w}$ | | w | $\bar{w}$ |
|---|---|---|---|---|---|
| $\theta_{AA'}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $S^{AB}$ | -1 | 0 |
| $\theta^{A}{}_{A'}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $S^{A}{}_{B}$ | 0 | 0 |
| $\theta_{A}{}^{A'}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $S_{AB}$ | 1 | 0 |
| $\theta^{AA'}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $S^{A'B'}$ | 0 | -1 |

Finally we note that since,

$$f^2 = \xi \text{ we have } L^{\tilde{A}}{}_{A} := \frac{1}{\sqrt{\xi}}\xi^{\tilde{A}}{}_{A} \text{ with } det(L^{\tilde{A}}{}_{A}) = 1, \tag{21}$$

we have then that (20) realises the isomorphism $SL(2,\mathbb{C})(2 \leftrightarrow 1)SO^{+}(1,3)$ by

$$T^{\tilde{a}}{}_{a} = -\frac{1}{2}\sigma^{\tilde{a}}{}_{\tilde{A}\tilde{A}'}L^{\tilde{A}}{}_{A}L^{\tilde{A}'}{}_{A'}\sigma^{AA'}_{a}, \tag{22}$$

## Appendix 3: Connection Potential Gauge Invariances

This work is distinct from that which has tried to relate Maxwell to linear connections which were non-metric or have torsion, [3]. We define the Ricci forms[30] according to

$$\gamma^{a}{}_{b} := \gamma^{a}{}_{bc}\theta^{c} = -\sigma^{a}{}_{\mu;\nu}\sigma_{b}{}^{\mu}dx^{\nu} \in \Lambda^{1}U.$$

Here we will use the soldering form and the $GL(2,\mathbb{C})$ connection to construct a real linear connection,

$$\gamma^{a}{}_{b} = \omega^{a}{}_{b} + \delta^{a}{}_{b}(A + \bar{A}). \tag{23}$$

Consider the pair of linearly independent contravariant spinors,

$$(k^{A}, l^{A}) \equiv (e^{A}{}_{\mathbf{1}}, e^{A}{}_{\mathbf{2}}) =: e^{A}{}_{\mathbf{A}}.$$

For $e := det(e^{A}{}_{\mathbf{A}}) = \frac{1}{2}\epsilon_{AB}e^{A}{}_{\mathbf{A}}e^{B}{}_{\mathbf{B}} \neq 0$ of density weight $(w,\bar{w}) = (1,0)$ we have for numerical Levi-Civita symbols that are not invariant under $GL(2,\mathbb{C})$,

$$\epsilon_{AB}e^{A}{}_{\mathbf{A}}e^{B}{}_{\mathbf{B}} = e\epsilon_{\mathbf{AB}},$$

from which we can define the covariant basis

$$e_{A}{}^{\mathbf{A}} = (e)^{-1}\epsilon_{AS}e^{S}{}_{\mathbf{S}}\epsilon^{\mathbf{AS}},$$

so that we have for example $e_{A}{}^{\mathbf{S}}e^{B}{}_{\mathbf{S}} = \delta^{B}{}_{A}$. We define our spinorial connection (in fact can be defined independent of basis) as;

$$\gamma^{A}{}_{B\mu} := -e^{A}{}_{\mathbf{S}}({}^{\gamma}\mathcal{D}_{\mu} - \partial_{\mu})e_{B}{}^{\mathbf{S}} = +e_{B}{}^{\mathbf{S}}({}^{\gamma}\mathcal{D}_{\mu} - \partial_{\mu})e^{A}{}_{\mathbf{S}}. \tag{24}$$

[30] We fix our definition of tensorial weight by considering the scalar components of tensorial density (of real weight $w_T$) with respect to a local co-ordinate map

$$T^{a\dots}_{b\dots} := [det(e^{m}{}_{\mu})]^{w_T}e^{a}{}_{\alpha}e_{b}{}^{\beta}\dots T^{\alpha\dots}_{\beta\dots},$$

stating that the tensor density of weight $w_T = 1$ is $\epsilon_{\alpha_1..\alpha_4}$ has scalar (numerical) components $\epsilon_{a_1..a_4}$.

Its general form is implicit in

$$
\begin{aligned}
{}^{\gamma}\mathcal{D}_\mu \psi^{AB'...}_{CD'...} &= \{\partial_\mu + w\gamma^S{}_{S\mu} + \bar{w}\gamma^{S'}{}_{S'\mu}\}\psi^{AB'...}_{CD'...} \\
&+ \gamma^A{}_{S\mu}\psi^{SB'...}_{CD'...} - \gamma^S{}_{C\mu}\psi^{AB'...}_{SD'...} + \gamma^{B'}{}_{S'\mu}\psi^{AS'...}_{CD'...} - \gamma^{S'}{}_{D'\mu}\psi^{AB'...}_{CS'...} + ... \qquad (25)
\end{aligned}
$$

As (24) transforms as a covariant vector we define $\gamma^A{}_B := \gamma^A{}_{B\mu}dx^\mu \in \Lambda^1 U$ transforming as

$$
\gamma^{\tilde{A}}{}_{\tilde{B}} = \xi^{\tilde{A}}{}_A(\xi^{-1})_{\tilde{B}}{}^B\gamma^A{}_B + \xi^{\tilde{A}}{}_S d(\xi^{-1})_{\tilde{B}}{}^S, \tag{26}
$$

which given that

$$
\xi^{\tilde{S}}{}_S d(\xi^{-1})_{\tilde{S}}{}^S = -dln(\xi) \quad \text{implies} \quad \gamma^{\tilde{S}}{}_{\tilde{S}} = \gamma^S{}_S - dln(\xi). \tag{27}
$$

Accordingly we can decompose our connection into symmetric and anti-symmetric parts

$$
\begin{aligned}
\gamma_{AB\mu} &= \gamma_{(AB)\mu} + \gamma_{[AB]\mu}, \\
&:= \Gamma_{AB\mu} + \frac{1}{2}\epsilon_{AB}\gamma_\mu.
\end{aligned} \tag{28}
$$

Due to symmetry of $\Gamma_{AB\mu}$ we have $\Gamma^S{}_{S\mu} = 0$ so that $\gamma^S{}_{S\mu} = \gamma_\mu$ and $\gamma^{S'}{}_{S'\mu} = \gamma'_\mu$ and according to respectively (27) and (26) we have the forms $\Gamma^A{}_B$ defined up to $SL(2,\mathbb{C})$ transformations induced by $GL(2,\mathbb{C})$ according to the affine representation,

$$
\tilde{\gamma} = \gamma - dln(\xi) \quad \text{and} \quad \Gamma^{\tilde{A}}{}_{\tilde{B}} = L^{\tilde{A}}{}_A(L^{-1})_{\tilde{B}}{}^B\Gamma^A{}_B(\epsilon) + L^{\tilde{A}}{}_S d(L^{-1})_{\tilde{B}}{}^S, \tag{29}
$$

where $L^A{}_B(\epsilon)$ belongs to $SL(2,\mathbb{C})$ and $L^A{}_B(0) = \delta^A{}_B$. Therefore $\gamma_\mu$ is a covariant vector defined up to some complex gauge and $\Gamma^A{}_B$ are defined up to $SL(2,\mathbb{C})$ induced by $GL(2,\mathbb{C})$ according to (21). We now elaborate (25) distinguishing ${}^{\gamma}\mathcal{D}$ from ${}^{\Gamma}\nabla$ according to

$$
\begin{aligned}
{}^{\gamma}\mathcal{D}_\mu \psi^{A_1..A_k B'_1..B'_l}_{C_1..C_p D'_1..D'_q} &= \{\partial_\mu + [w + \frac{1}{2}(k-p)]\gamma_\mu + [\bar{w} + \frac{1}{2}(l-q)]\gamma_\mu\}\psi^{A_1..B'_1..}_{C_1..D'_1..} \\
&\quad + \Gamma^{A_1}{}_{S\mu}\psi^{S...B'_1..}_{C_1..D'_1...} - \Gamma^S{}_{C_1\mu}\psi^{A_1...B'_1..}_{S...D'_1...} + ... \\
&= \{{}^{\Gamma}\nabla_\mu + [w + \frac{1}{2}(k-p)]\gamma_\mu + [\bar{w} + \frac{1}{2}(l-q)]\gamma_\mu\}\psi^{A_1..B'_1..}_{C_1..D'_1..}
\end{aligned} \tag{30}
$$

and identify $\gamma_\mu$ with the complex electromagnetic potential[31], $\mathcal{A} = \breve{\mathrm{A}} + \mathring{\mathrm{A}}$ written in terms of real $\breve{\mathrm{A}}$ and pure imaginary $\mathring{\mathrm{A}}$ parts by noting the following identification

$$
\gamma_\mu := -\frac{ie_0}{2\hbar}\mathcal{A}_\mu = -\frac{ie_0}{2\hbar}(\breve{\mathrm{A}}_\mu + \mathring{\mathrm{A}})_\mu. \tag{31}
$$

[31] Maxwell's equations are written as,

$$
{}^*d\mathbf{f} = (4\pi i)\mathbf{j} \text{ or equivalently } \delta\mathbf{f} = 4\pi\mathbf{j} \text{ for co-differential, } \delta := -i^*d^*
$$

The self-dual complex 2-form Faraday field, ${}^*\mathbf{f} = \mathbf{f}$ is harmonic (so that $\mathbf{f} = -d\mathcal{A} \in \Lambda^2$) in the sense that it satisfies both $d\mathbf{f} = 0$ and $\delta\mathbf{f} = 0$.

We have then the $GL(2,\mathbb{C})$ decomposition of the connection of dimension[32] $(length)^{-1}$

$$\gamma^A{}_B = \Gamma^A{}_B + \delta^A{}_B A, \text{ for complex } A := -\frac{ie_0}{2\hbar}\mathcal{A}.$$

If one does not assume that $A$ is $u(1)$-valued the real part is non-zero so the real linear connection, (23) will not be metric. The reality of the Maxwell field arises when restricting the gauge group, $GL(2,\mathbb{C})$ to $SL(2,\mathbb{C}) \otimes U(1)$ as the latter term in (23) then vanishes when $A$ is required to be $u(1)$-valued and the Maurer-Cartan form is pure imaginary for real scalar potential $\Lambda$ as

$$A(\Lambda) = e^{-i\Lambda} de^{i\Lambda} = id\Lambda =: i\breve{\mathrm{A}} \text{ and } \bar{A} = -i\breve{\mathrm{A}}$$

Interestingly, as we will see with fermionic coupling, this becomes a necessary condition when charged fermionic matter as added according to the prescription of [7] and [8]. If we had rather applied the natural axioms of a covariant exterior derivative such as

$$\begin{aligned} {}^{\gamma}D\epsilon_{AB} &= -Q\epsilon_{AB}, \\ {}^{\gamma}D\epsilon^{AB} &= Q\epsilon^{AB}, \end{aligned}$$

for $Q$ a spin-weightless complex covariant vector, so that

$${}^{\gamma}D\epsilon_{AB} = D\epsilon_{AB} - 2\epsilon_{AB}A, \text{ for } A = \frac{1}{2}Q.$$

## Appendix 4: Semi-Chiral Lagrangian for Fermions

The Lagrangian for fermionic matter has a dependence on the connection so admits Torsion contributions but nevertheless can be written as the sum of a semi-chiral complex Lagrangian for vacuum General Relativity, $L_{SC}(\theta,\Gamma)$, a complex (semi)chiral fermionic matter Lagrangian, $L_{\frac{1}{2}}$ and a term, $L_{J^2}$ that ensures the standard Einstein-Weyl form of the field equations.

$$\begin{aligned} L_{SC}(\theta,\Gamma) &= i\theta^A{}_{A'} \wedge \theta^{BA'} \wedge F_{AB}, \\ L_{\frac{1}{2}}(\theta,\Gamma,\lambda,\tilde{\lambda}) &= +\eta^{AA'} \wedge \tilde{\lambda}_{A'}\lambda_A, \\ L_{J^2}(\lambda,\tilde{\lambda}) &= \frac{3}{16}\lambda_A\tilde{\lambda}_{A'}\lambda^A\tilde{\lambda}^{A'}, \\ L_{Tot} &= L_{SC} + L_{\frac{1}{2}} + L_{J^2}. \end{aligned}$$

The $\lambda_A(\tilde{\lambda}_{A'})$ are the left (resp. right)-handed zero forms. The theory uses only the anti-self dual connection, $D$ (which does not act on tensors, so for example,

$$D\theta^{AA'} = d\theta^{AA'} - \theta^{BA'} \wedge \Gamma^A{}_B,$$

but is complete and it turns out, (by varying $K^A{}_B$), that the real source current, $J_{AA'} = \lambda_A\lambda_{A'} = -J_{A'A}$ supports only the axial part of the torsion of $\Gamma^A{}_B$, written in terms of the contorsion form as,

[32] $e_0$ is a real constant of dimension charge and the dimensions of the potential, $\mathcal{A}_\mu$ are $\frac{e_0}{length}$ so that $dim(\mathcal{A}_\mu) = -\frac{i}{2}\frac{1}{137}\frac{1}{length}$.

$$K_{AB} = -\frac{1}{4} J_{C'(A}\theta_{B)}{}^{C'}.$$

Because ultimately the real theory is of interest (where $\tilde{\lambda}_{A'} = \overline{\lambda_A}$ and $\theta$ is hermitian) it proves useful to extend $D$ to $\nabla$. Although it is argued that the spin $\frac{1}{2}$ field variables can be taken to be either Grassman [or complex]-valued, in fact the use of complex spin $\frac{1}{2}$ fields leads to a non-standard energy-momentum tensor which includes quartic spin $\frac{1}{2}$ fields.

## Appendix 5: Decomposition of Generalised Weyl Curvature Spinor

The spinor with the interchange symmetry $\Xi_{ABCD} = \Xi_{CDAB}$ used as a multiplier field in the $GL(2,\mathbb{C})$-form formulation of complex Einstein-Maxwell, [14] has a partial decomposition of

$$\Xi_{ABCD} = \Xi_{(AB)(CD)} + \frac{1}{2}\{\epsilon_{AB}\Xi_E{}^E{}_{(CD)} + \Xi_{(AB)E}{}^E\epsilon_{CD}\} + \frac{1}{4}\epsilon_{AB}\epsilon_{CD}\Xi_E{}^E{}_F{}^F. \tag{32}$$

The spinor $a_{ABCD} := \Xi_{(AB)(CD)}$ possesses the symmetries of $X_{ABCD}$, the curvature spinor [24] for an $sl(2,\mathbb{C})$-valued connection with Torsion and Cosmological constant term,

$$X_{ABCD} = \Psi_{ABCD} + (\epsilon_{BC}\epsilon_{AD} + \epsilon_{BD}\epsilon_{AC})\Lambda + (\epsilon_{CB}\chi_{AD} + \epsilon_{BA}\chi_{CD} + \epsilon_{DB}\chi_{AC}). \tag{33}$$

With the additional interchange symmetry $a_{ABCD} = a_{CDAB}$ (that arises from the even number Grassman structure of the bivectors of space-time) this means that its Torsion contributions are zero, $\chi_{AD} := \frac{1}{6}a_{H(AD)}{}^H = 0$ so that $Xi_{ABCD}$ decomposes as,

$$\begin{aligned}
&\Xi_{ABCD} = \Psi_{ABCD} + 2\epsilon_{B(C}\epsilon_{|A|D)}\lambda + \epsilon_{AB}\phi_{CD} + \phi_{AB}\epsilon_{CD} + 2\epsilon_{AB}\epsilon_{CD}k, \text{ with} \\
&\Psi_{ABCD} := a_{(ABCD)}, \quad \lambda := \frac{1}{6}\Xi^{HE}{}_{EH}, \quad k := \frac{1}{8}\Xi_E{}^E{}_F{}^F, \quad \phi_{AB} := \Xi_{(AB)E}{}^E.
\end{aligned}$$

Given further, $\Xi_E{}^E{}_F{}^F = -\frac{2}{3}\Xi^{AB}{}_{AB}$, we have that

$$\Xi_{ABCD} = \Psi_{ABCD} + 2\epsilon_{B(C}\epsilon_{|A|D)}\lambda + \epsilon_{AB}\phi_{CD} + \phi_{AB}\epsilon_{CD}. \tag{34}$$

The use of self-dual gauge freedom is used and $k$ is chosen to be $-\frac{1}{2}$ then $\bar{\phi}_{A'B'}$ is the complex conjugate of $\phi_{AB}$ and the Faraday field, $\mathbf{f}$ is real. If in addition we insist that

$$\Xi^A{}_A{}^B{}_B = \Psi^A{}_A{}^B{}_B + 2\epsilon_A{}^B\epsilon^A{}_B\lambda + \epsilon^A{}_A\phi^C{}_C + \phi^A{}_A\epsilon^B{}_B = \Xi^A{}_B{}^B{}_A$$

then we arrive at the equations of motion with $\lambda = 0$.

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