Torsional twists of inertial Dark Matter-Energy

Lee McCulloch-James
Mathematics Department*, Dwight school, Barnet, London

December 18, 2019

Abstract

We review of the some basic notions of the quantity and quality of energy and then apply energy concepts to Dark (non Baryonic) matter to ask whether Nature's Book keeper tracks inertial mass-energy entries above and below her Energy Conservation line. We then outline a complex spinor version of the non-local Teleparallel theory for gravity based on Cartan's notion of Torsion of Hehl and Bashhoon, [3]

Keywords: Dark Matter, Torsion, Teleparallel theory

Einstein made explicit the implicit inertial and gravitational mass equivalence of Newton in unifying Heavenly motions to those of the inertial ones on Earth. Is there a

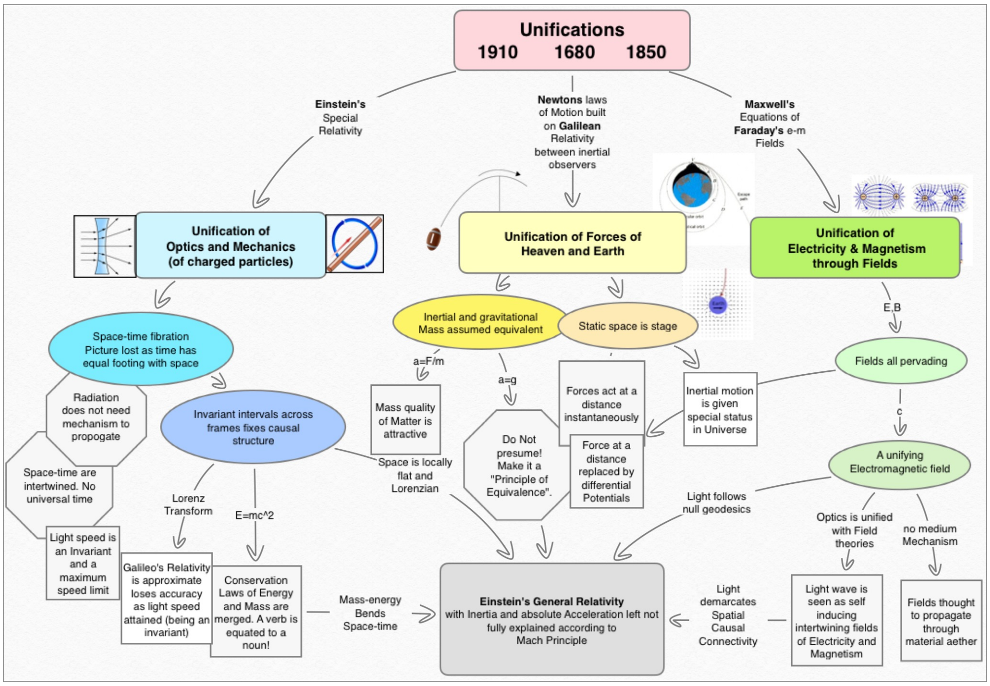


Figure 1: Unification is so twentieth century

kindred assumption waiting to be unpicked of Dark Matter, ${}_Dm_0$? We do not darken the Energy equivalent of *Dark Matter* because the *Dark Energy* term is already assigned

*https://mcmurmerings.wordpress.com/, leemcjames@gmail.com

to the (Cosmological Constant) stuff of the vacuum. Need it be so that inertial Dark Matter, ${}_{D}m_0$ has an equivalent Dark Mass-Energy, ${}_{D}E/c^2$? Precedent is there for asking if we are making a misplaced presumption: Einstein asked of Newton's Law of Gravitation, whether the inertial mass of the Baryonic test (mass) material, ${}_{B}m_0$ can be equated to its gravitational mass ${}_{B}m_G$.

A certain quantity, which may be determined by experiment, must remain constant. This quantity is the sum of two terms [potential energy, U and Kinetic energy, T].

Energy of Conserved sums and Optimised differences

From Poincare, the epigraph above we note that *quantity* is more formal than amount or number and it is indeed a noun but the certain is anything but. He elaborates, [9]

"The first depends only on the position of the material points, and is independent of their velocities; the second is proportional to the squares of these velocities. This decomposition can only take place one way. [Only the function] T +U which we call Energy maintains this independence.Every change that the bodies of nature can undergo is regulated by two experimental laws. First, the sum of the kinetic and potential energies, T+U is constant. This is the principle of conservation of energy. "

But maybe less so when articulated as the (Lagrangian) difference. That is above and below the line of the ledger book, that Nature apparently teleologically always seeks to minimize as a system evolves.

"if a system of bodies is at A at the time time t_A and at B at tme t_B, it always passes from the first position to the second by such a path that the mean value of the the difference of the two, T-U in the interval which separates the two epochs is a minimum. This is Hamilton's principle, and is one of the forms of the principle of least action."

The Least action principle explains why light traversing media of different optical densities at some angle bends to follow an elephant's path of minimal elapsed traversal time. That timelessness is wasted on a photon-as youth is on the young- the illusory null frame in which photons would notionally be stationary, time does not tick. We can countenance perhaps a conservation law of energy flux in which for every black entry there is a red. But for Nature to optimise a difference is deemed too much to put to secondary school students. Preposterous, that Nature apparently traces all paths: the one (in actuality) realised in addition to the infinite multitude of inactual traced counterfactual paths. All to find before the fact the one that minimes the photon's Action expenditure. Solace of sorts is to be found in the explanation that is the path integral approach of Feynman.

Of Dark Energy and Dark Matter-Energy

What of the mass-energy of stuff itself and the wastelands of the in-between? What is to be said of the conservation principle of Baryonic matter-energy, ${}_{B}E = \gamma m_0 c^2$?

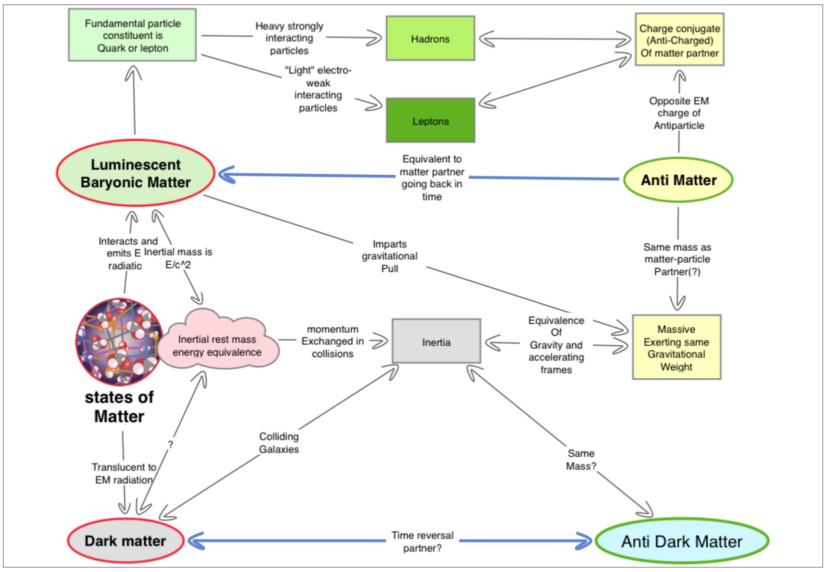


Figure 2: Gravitational and Inertial mass-energy Equivalence of Baryonic and Dark Matter and (possible) Ant-particle partners.

For an inertial (rest) mass ${}_Bm_0$ of Baryonic matter we have our energy equivalent, ${}_BE/c^2$. Energy interchanged with noun stuff so we arrive at its corporeal form. Poincare,[9] says:

“..by what right do we apply to the ether the mechanical properties observed in ordinary matter, which is but false matter? The ancient fluids, caloric, electricity , were abandoned when it was seen the heat is not indestructible.”

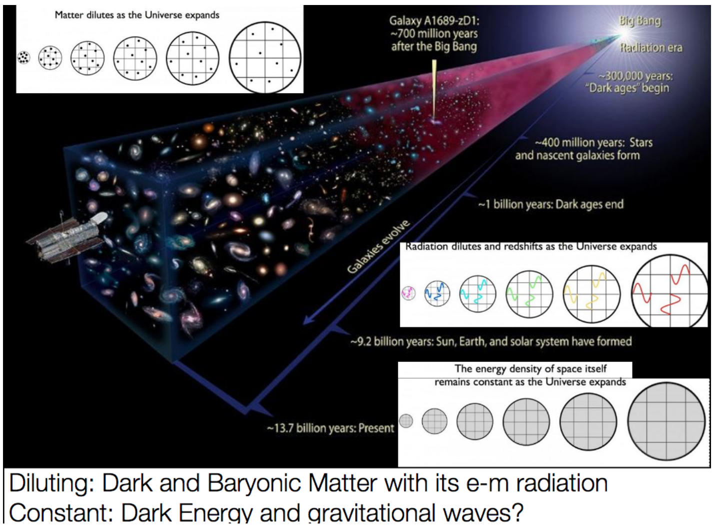


Figure 3: Conservation of Energy-Density

Our Baryonic mass centric perspective is ego-centric, not least because most of the mass-energy in the universe is not of our Baryonic type and that the all-pervading Cosmic Microwave Background Radiation (CMBR) that once dominated our universe will do so again in the future. Whilst the density of our such stuff and that which it radiates dilutes as the universe expands, the energy-density of the space between stuff remains a constant.

Some reminders of what we think we know. We remind ourselves that only time ticks for that part of mass-energy that possesses inertia as such the CMBR is not getting old in its null frame of reference only from our leaden perspective. We can

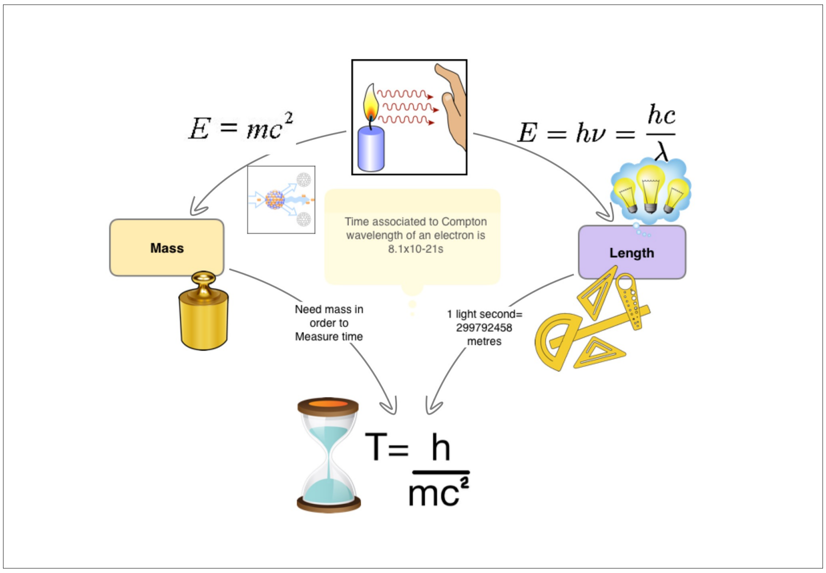


Figure 4: Time does not tick for massless photons

only wistfully gaze at (and with) those time optimisers that are photon, from our decaying leaden sub-luminal frame of reference. Only when there is vibrating inertial mass does the Bookkeepers ledger book mark down energy for depreciation. We remind ourselves of what all this CMBR and latterly emitted Solar thermal energy actually is by most easily by identifying what it is not. Of all the decay phenomena in the universe the least thermal, as Hawking and Beckenstein may argue, in an increasing its Entropy sense at least, seems to be the generation of gravitational waves by Black Hole/Neutron star merger ring-downs.

Energy of the in-between and the Dark beyond

The twenty-first conception of the vacuum that resides between stuff is becoming more of a some-*thing*, if not required to mechanically propagate the exchange particles that give rise to what we observe as interaction. The ether thing that Poincare describes below is what we presently ascribe the name of Dark Energy, conceived as a substrate of broiling virtual particles of an unaccountably large mass-energy. Poincare,[9] says:

"We may conceive of ordinary matter as either composed of atoms whose internal movements escape us []; or we imagine one of those subtle fluids, which under the name of ether or other names, have from time to time played so important a role in physical theories. Often we go further, and regard the ether as the only primitive, or even as the only true matter. "

We note now how far we have come by quoting again Poincare, [9]

"[Moderates] see in matter nothing more than the geometrical locus of singularities in the ether. Lord Kelvin [regards] matter to be the only locus of those points at which the ether is animated by vortex motions. Riemann believes [matter] to be the locus of points at which ether is constantly destroyed; to Wiechart or Larmor, it is the locus of the points at which the ether has undergone a kind of Torsion."

We have perhaps not come so far, from material arising from the torsional twists of vortices to their emergence from knots of loops of strings, the irreducible stuff that the bookkeeper quantifies escapes our qualification.

Inertial mass-energy of Dark Matter

We do not know what the inertia of matter is but we should follow Einstein in equating it -through his Equivalence Principle -to the gravitational binding, negatively Entropic qualities of matter. We may ask why would the unification of electrodynamics and light-optics as embodied in the mechanics of Einstein's Special Relativity put constraints on a (*Dark*) matter type that itself does not respect the seeing power of Baryonic induced light?

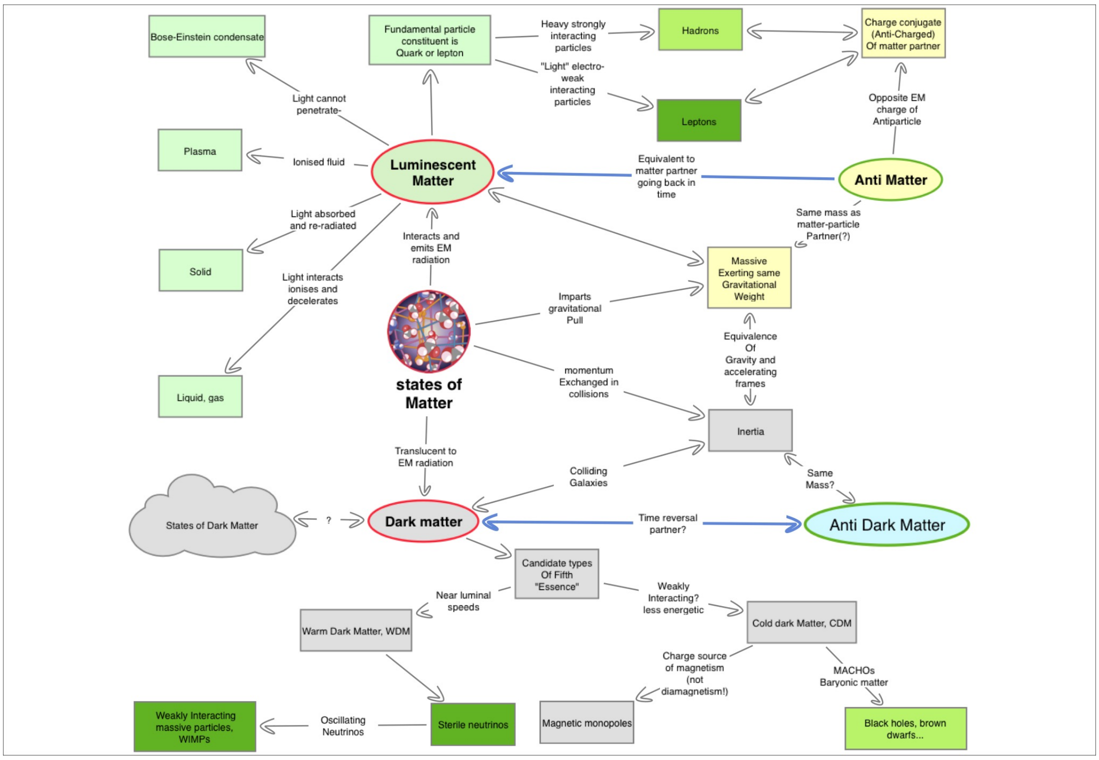


Figure 5: We barely know about the states of all matter let alone their energy content.

We do not know the WIMPy, Axion-like or otherwise nature of the gravitationally interacting Dark Matter of the universe. We do not know, but the working presumption is, that it is bound by the two Einsteinian equivalences of our Baryonic materiality. Einstein asked why the inertial mass of Baryonic material, ${}_Bm_0$ can be equated to its gravitational mass ${}_Bm_G$. Einstein elevated to a principle this equivalence of the gravitational free fall field of the Heavens and inertial acceleration of Earthly objects. The free falling g-field exists independent of whether there is a corporeal test mass to fall under its potential. Is there a similar presumption waiting to be unpicked of the Dark Matter inertial test mass, ${}_Dm_0$?

Equivalence Principles

Before the technical part to follow we finish with some notes on the Equivalence Principle. We do not Darken the Energy equivalent of Dark matter because that *Dark Energy* term is already assigned to the negative pressure of the expanding Cosmological vacuum. It must be surely so that inertial Dark Matter, ${}_Dm_0$ has an equivalent Dark Mass-Energy, ${}_DE/c^2$. But why would the unification of electrodynamics and light-optics embodied in the mechanics of Einstein's Special Relativity put constraints on a (dark) matter type that does not respect the seeing power of Baryonic induced light? As anti-Entropic Baryonic material, so indebted to our illuminating photon source the Sun, we humans are perhaps guilty of assigning to it a misplaced omnipotence.

We note that a photon has momentum and thus traces null geodesics that are influenced by the gravitational metric field, g_{ab} or potential $\Gamma^a{}_b$. We note also that there are Dark Matter candidates *dark photons* that do interact with *light photons*. Aslo, as we will need this later, we recall that while Noether symmetry theorems (that give rise to energy-momentum conservation laws) do not care about the Equivalence Principle, delivering currents in any coordinate system, the presence of an additional boundary term in a Lagrangian for your theory of gravitation will necessarily change the value of that Lagrangian's (Noether) conserved charges (energy-momentum currents).

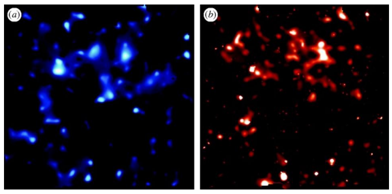
Consider the three increasingly restrictive Equivalence Principles (EP), Weak, Einsteinian, and Strong (SEP). Strong EP alone applies to self-gravitating objects (such as stars) which have substantial internal gravitational interactions and has two requirements:

weak EP version applies to objects (stars etc) exerting a gravitational force on themselves such that the gravitational motion of a small test body depends only on its initial position in spacetime and velocity, and not on its constitution;

Einstein EP restated to allow for self-gravitating bodies so that the outcome of any local experiment (gravitational or not) in a freely falling laboratory is independent of the velocity of the laboratory and its location in spacetime.

The Stronger Einstein Equivalence Principle (SEEP) says that Einstein relativised accelerations not the velocities in the tangent space to M_4. SEEP requires that the gravitational constant, G be the same everywhere in the universe and the effect of gravity on a body does not depend on the nature of the mass-energy or internal structure of that body.

(a) The Bullet Cluster exhibits a separation of X-ray gas from its inferred gravitational signal.



(b) inferred density of dark matter as inferred vs map for the baryonic matter

Figure 6: R. Ellis, Philos Trans A Math Phys Eng Sci. 2010 Mar 13; 368(1914)

Non-local Torsional Twists

Inference from Bullet Cluster

Non inertial effects are suggested by Siegel, ([5]) as an explanation for the misalignment of optical and gravitationally inferred sources.

"When your cluster is undisturbed, the gravitational effects are located where the matter is distributed. It's only after a collision or interaction has taken place that we see what appears to be a non-local effect. This indicates that something happens during the collision process to separate normal matter from where we see the gravitational effects."[5]

One way of explaining this differential in the distribution of Baryonic and inferred gravitation is by a non-local argument. Massoon and Hehl, [3] have developed this framework which I will outline emphasising the teleparallel and constitutive equation aspects of their argument.

Complex Framework for non-local Teleparallism

Geometrically Einstein's freely falling elevator is captured in the soldering functor $\mathbf{e^a}_\mu$. In providing the local identification of space-time, μ with local Lorentz quantities, i of the observer within the definition of the non exact differential form, $\theta^a = \mathbf{e}^a{}_\mu dx^\mu$, it cogently encodes his Equivalence Principle of inertial and gravitational masses. These co-frames being not integrable at each event constitute a non-coordinate (anholonomic) Lorentz basis, (i).[1] The introduction of a Riemannian, g or symplectic , ϵ metric gives rise to a natural isomorphism between the tangent space and the cotangent space at a

[1] Such a co-frame formulation of General Relativity is a Cartan G-structure with cotangent bundle, T*B soldered to spacetime: for a configuration space of a generic field, ϕ^A the fibre bundle of frames is $\pi : B \to M$. While the cotangent bundle T*B of the symplectic geometry of phase space does not need a metric structure on the basic world sheet, M to define the differential of a function, a metric is needed in order to define the gradient on its (dual) tangent bundle, TB. As such the tangent co-vector of T*B is called the canonical one-form or symplectic potential and can be viewed as that primitive object from which the metric structure on the base is derived.The view is that the internal indices AA' are associated to some (subset of) $GL(2,\mathbb{C})$ spin structure over space-time become spinor indices through the dynamical soldering form, $\theta^{AA'}{}_\mu$. Both the TB and the T*B at a point are both real vector spaces, V and U of the same dimension and therefore isomorphic to each other via many possible isomorphisms. That space-time, M itself affords a Geometric Algebra can be summarised by

point, associating to any tangent co-vector a canonical tangent vector. This is implicit[2] in the defining of an oriented $\mathbb{C}$-valued version of a 4-volume pseudo-scalar $\eta \in \Lambda^4 U$ of space-time through the use of the Hodge dual operation *,

$$\eta = \ ^*(\theta^a \wedge \theta^b \wedge \theta^c \wedge \theta^d) := \frac{1}{4!}\phi\tilde{\epsilon}_{abcd}\theta^a \wedge \theta^b \wedge \theta^c \wedge \theta^d$$
$$\phi\tilde{\epsilon}_{abcd} \leftrightarrow i(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'}).$$

In standard metric-affine formulations the space–time metric emerges from the more fundamental metric symplectic spinors according to $g_{ab} \leftrightarrow \epsilon_{AB}\epsilon_{A'B'}$. Typically ϕ would be the square root of the modulus of the metric determinant, $\sqrt{g}$ a tensor density of weight +1 to offset the -1 weight of $\tilde{\epsilon}$.

Given a metric tensor structure on a small enough locale of space, M_4 the inner product of two vectors within M_4's tangent space can be computed directly. The tetrad (frame) field, $\mathbf{e}_a$ is that linear map from the tangent space, $(\mu = 0, 1, 2, 3)$ to Minkowski space that preserves this inner product being a set of four $a = (0, i|i = 1, 2, 3))$ orthonormal vector fields, one timelike, $\mathbf{e}_0 = \mathbf{e}_t$ and three space-like, $\mathbf{e}_i$ defined on a Lorentzian manifold whose integral curves are the world-lines of observers.

An observed event on the worldline, is measured with respect to the triad of three space-like unit fields that define a local laboratory frame, L. The frame fields, $\mathbf{e}_a$ can be regarded as the "matrix square root" of the metric tensor as, $g_{\alpha\beta} = \mathbf{e}^a{}_\alpha \mathbf{e}^b{}_\beta \eta_{ab}$. Tensorial quantities defined on the manifold can be expressed either using the frame field, $\mathbf{e}_a$ on TM or its dual co-frame field, θ^a on T*M. The linear approximation for any gauge theory of gravity, [6] is defined by $\mathbf{e}^a{}_\mu = \delta^a{}_\mu - \psi^a{}_\mu$ and $\mathbf{e}^\mu{}_a = \delta^\mu{}_a + \psi^\mu{}_a$ in which ψ is assumed small and as such there is no distinction between holonomic co-ordinate indices, μ and Lorentzian a. As such we can say $g_{ab} = \eta_{ab} + h_{ab}$ for $h_{ab} := 2\psi_{(ab)}$. In co-frame language this reads $\delta\theta^a \equiv \dot{\theta}^a = -\psi^a$. More generally, ψ be a basic field, (Λ a matrix representation of the Lorentz group) in the global background inertial frame and $\hat{\psi} = \Lambda\psi$ the field measured instantaneously by the infinite set of hypothetical momentarily comoving inertial observers along the world line of an accelerated observer. Mashhoon, [6] points out,

"In the standard theory of special relativity, Lorentz invariance is extended to accelerated observers in a pointwise manner via the hypothesis of locality [as an] accelerated observer is assumed to be pointwise inertial [so] that the proper time τ is in fact the time as determined by the accelerated observer (clock hypothesis)."

In Newtonian mechanics, the state of a point particle is characterized only by its position and velocity. For classical point particles and rays of radiation instantaneous

considering the aggregate of its multi-vector structures,

[scalars, vectors, bivectors, trivectors, pseudoscalar-volume form]

on the linear (co)-tangent spaces U,V to space-time, M. Given U we can construct the exterior space ΛU, the space of aggregates of multivectors of these different orders (0 to 4) of U that is closed under the exterior $\wedge$ product. In this way ΛU becomes an algebra, the Grassman or exterior algebra.

[2] $\leftrightarrow$ idicates isomorphism between Levi-Cevita tensor density and symplectic spinor metric.

measurements are in principle possible. The Newtonian limit corresponds to the assumption that $\psi_{00} = \psi_{aa} = \frac{1}{c^2}\Phi, \psi^a{}_a = \frac{2\Phi}{c^2}$ where Φ is the Newtonian gravitational potential whose off-diagonal components vanish.

Non-local Constitutive equations

As with Poincare's definition of the Lagrangian it is the averages of fields that matter, Mashhoon, [6]

"Basic field measurements [] cannot be performed instantaneously [as] according to Bohr and Rosenfeld, the electric field $E(t, x)$ and magnetic fields $B(t, x)$ occur in Maxwell's equations as idealizations; only the spacetime averages of these fields have immediate physical significance".

For a field $\hat{\Psi}(\tau)$ actually measured by the accelerated observer we are looking for a non-local generalisation of the Lorentz (transformation) relationship between $\hat{\Psi}(\tau)$ and $\hat{\psi}$. We have then that non-local Special Relativity involves local fields $\hat{\psi}(x)$ in Minkowski spacetime that satisfy *integro-differential* field equations, [6]

$$\hat{\Psi}(\tau) = \hat{\psi}(\tau) + u(\tau - \tau_0) \int_{\tau_0}^{\tau} -\frac{d\Lambda(\tau')}{d\tau'} \Lambda^{-1}(\tau') \hat{\psi}(\tau') d\tau',$$

for unit step function such that $u(t) = 0$ for $t < 0$ and $u(t) = 1$ for $t > 0$, and τ_0 is the instant that acceleration is turned on up until which the Kernel of the integral is zero. We have then, [6]

"What is measured at proper time τ (by the accelerated observer) is the field $\hat{\psi}(\tau)$ measured by the instantaneously comoving inertial observer at τ together with a certain average over the observer's past world line that constitutes the linear memory of past acceleration."

Susceptibility of Hodge Duality

In the phenomenology of electrodynamics of media, the *state* constitutive relations are typically nonlocal. A reminder of the former of the these equations in Cartan's differential form formalism follows. In our present material universe, of the four Maxwell equations only Faraday's and modified Ampere's laws are independent. Free charges and currents are the sources for the four electric and magnetic (resp.) fields and their fluxes E, D and H, B which can be neatly gathered up as tensor-valued two forms, F and G.

The constraining system of equations that describe the behaviour of matter under the influence of these fields, known as the Constitutive ("having the power to establish or give organised existence to something") relations. In the presence of external electric (magnetic) fields any permeated "substrate" susceptible medium becomes polarized (magnetized) such that the electric flux density of the medium is characterised by a polarisation vector indicating the dipole moment per unit volume as the displacement current, $D = \epsilon_0 E + P$. Similarly the magnetic flux density in a magnetic medium is $B = \mu_0 H + M$ for magnetisation vector, M captures the magnetic dipole moment per unit volume.

In the absence of any material (i.e. in a vacuum) the Constitutive relations are given by $D = \epsilon_0 E, B = \mu_0 H$ with an invariant form written using the Hodge dual, $G = \chi(^\star)F =^\star F$. The final equality being true because the vacuum susceptibility is assumed to be trivial so that Electric, Magnetic fields and fluxes are wholly intertwined by the conformally invariant Hodge dual operator.

Twisted frames of Teleparallellism

Even with flatness, twistiness can exist. With a metric connection, non-zero torsion vectors (perpendicular to the tangent vector of a curve) will rotate around it like a corkscrewing helix, Cartan, 1922, [3]

i A vector which is parallel transported along itself does not change, so a vector both directed and transported in x-direction.

ii A vector that is orthogonal to the direction of transport rotates with a prescribed constant 'velocity" so a vector in y–direction transported in x–direction).

"Imagine a space F which corresponds point by point with a Euclidean space E, the correspondence preserving distances. The difference between the two spaces is the following: two orthogonal triads issuing from two points A and A' infinitesimally nearby in F will be parallel when the corresponding triads in E may be deduced one from the other by a given helicoidal displacement (of right– handed sense, for example), having as its axis the line joining the origins. The straight lines in F thus correspond to the straight lines in E: They are geodesics. The space F thus defined admits a six parameter group of transformations; it would be our ordinary space as viewed by observers whose perceptions have been twisted. Mechanically, it corresponds to a medium having constant pressure and constant internal torque."

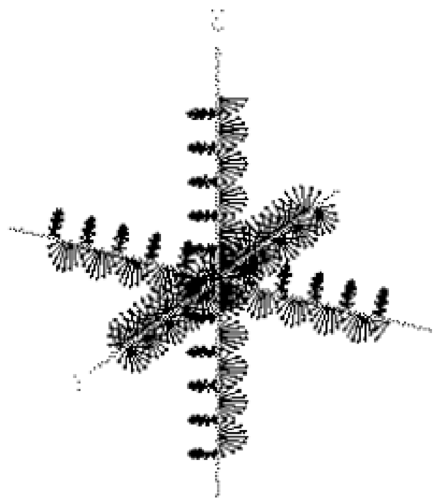


Figure 7: Cartan's depiction of twisting Frame fields

conTorsion of Cartan Structure Equations

With $\Gamma^A{}_B$ and $\bar{\Gamma}^{A'}{}_{B'}$ (complex conjugate) $sl(2,\mathbb{C})$-valued connection one-forms and the torsion two form denoted as $\Theta^{AA'}$, the first Cartan structure equation reads

$$\Theta^{AA'} := d\theta^{AA'} - \theta_{AB'} \wedge \bar{\Gamma}^{A'}{}_{B'} - \theta_{BA'} \wedge \Gamma^A{}_B.$$

That is for Torsion two form, $\Theta^{AA'} := \nabla\theta_{AA'}$, where $\nabla \equiv {}^\Gamma\nabla$ denotes the exterior covariant derivative with respect to the $sl(2,\mathbb{C})$-valued metric connection(s) acting linearly on the soldering form[3] of space M. Defining the basis of anti-self dual two-forms[4] as $\Sigma^{AB} := \frac{1}{2}\theta^A{}_{A'} \wedge \theta^{BA'}$, the second Cartan structure equations take the complex form

$$F^A_B := d\Gamma^A_B + \Gamma^A_C \wedge \Gamma^C_B,$$

$$F^A_B := \Psi^A_{BCD}\Sigma^{CD} + \Phi^A_{BC'D'}\bar{\Sigma}^{C'D'} + 2\Lambda\Sigma^A_B + (\chi_D{}^A\Sigma_B{}^D + \chi_{DB}\Sigma^{AD}),$$

where the curvature two-form, F^A_B, has been decomposed into spinor fields of dimension 5,9,1 and 3 respectively , corresponding to the anti-self dual part of the Weyl conformal spinor, Ψ^A_{BCD}, the spinor representation of the trace-free part of the Ricci tensor, $-2\Phi^A_{BC'D'}$ and the Ricci scalar 24Λ, - all with respect to the curvature of the $SL(2,\mathbb{C})$ connection and χ^{AB} arising from the presence of non-zero torsion. The breakdown of the spinor form of the $SL(2,\mathbb{C})$ connection and its contorsion parts follows Penrose, Mielke [2] although different conventions are used,

$$\Theta^a = \frac{1}{2}\Theta^a{}_{bc}\theta^b \wedge \theta^c = d\theta^a + \Gamma^a{}_b \wedge \theta^b,$$

$$\Theta^a{}_{bc}e_a = ({}^\Gamma\nabla_c e_b - {}^\Gamma\nabla_b e_c) = (-\Gamma^a{}_{bc} + \Gamma^a{}_{cb})e_a.$$

The spinorial decomposition of the (vector-valued) Torsion 2-form reads,

$$\Theta_a \leftrightarrow \Theta_{AA'BC}\Sigma^{BC} + \bar{\Theta}_{AA'B'C'}\bar{\Sigma}^{B'C'}\Theta_{ABCA'},$$

where the spinor $\Theta_{ABCA'}$ decomposes respectively into totally symmetric and 'axial', $\hat{\Theta}_{CA'}$ parts as,

$$\Theta_{AA'BC} := \frac{1}{2}\Theta^{P'}_{ABCA'P'} = \sigma_{ABCA'} + 2\epsilon_{A(B}\hat{\Theta}_{C)A'} = \Theta_{A(BC)A'},$$

$$\sigma_{ABCA'} := \frac{1}{2}\Theta_{(A|A'|C|P'|B)}{}^{P'},$$

[3]The internal 'symplectic metric', ϵ_{AB} is given as fixed so that the internal $SL(2,\mathbb{C})$ connection is then traceless $\Gamma_{AB} = \Gamma_{BA}$ due to $\nabla\epsilon_{AB} = 0$.

[4]Here we have used the basis of anti-self dual two-forms Σ^{AB}, defined in terms of the co-frame dynamical variable for mere ease of exposition. In this spirit, the variational formulation of Plebanski[7] uses * as an indempotent algebraic structure splitting the local algebra of the complexified $SO(1,3)_{\mathbb{C}} \xrightarrow{\sim} SO(4,\mathbb{C})$ gauge bundle over space-time, M into left and right handed ideals,

$$so(1,3)_{\mathbb{C}} = so(1,3\mathbb{C})^+ \oplus so(1,3\mathbb{C})^-.$$

The fully chiral dynamical variable, a basis of anti-self dual two-forms Σ^{AB}, is to be interpreted as the gauge-potential field which in the weak field limit possesses an excited "graviton" state.That is, once reality conditions are applied "off-shell" to an otherwise complex field.

$$\hat{\Theta}_{CA'} := -\frac{1}{6}\Theta_{EA'CP'}{}^{EP'} = \frac{1}{3}\Theta^{D}{}_{CDA'},$$

which transform according to the $(\frac{3}{2},\frac{1}{2})$ and $(\frac{1}{2},\frac{1}{2})$ representations. Here (i,j) denotes the finite dimensional representations of $sl(2,\mathbb{C}))]$ with dimensions 8 and 4 respectively. Consider now, two metric compatible exterior covariant derivatives ${}^{\Gamma}\nabla$ and ${}^{\omega}\nabla$ associated to the metric connections, Γ_{ab} and ω_{ab} (${}^{\omega}\Theta^{a} = 0$). The difference of their action on a vector U^{b} is given in terms of spinors as,

$$({}^{\Gamma}\nabla_{a} -{}^{\omega} \nabla_{a})U^{BB'} = U^{CB'}K^{B}{}_{Ca} + U^{BC'}\tilde{K}^{B'}{}_{C'a},$$
$$U^{BB'} = U^{CC'}(\epsilon_{C'}{}^{B'}K^{B}{}_{CAA'} + \epsilon_{C}{}^{B}\tilde{K}^{B'}{}_{C'AA'}),$$

so that upon adopting the useful notation $\delta^{A}{}_{B} := \epsilon^{A}{}_{B}$ the contorsion tensor, McRae,[?] has the spinor form

$$K^{b}{}_{ca} \leftrightarrow \delta^{B'}{}_{C'}K^{B}{}_{CAA'} + \delta^{B}{}_{C}\tilde{K}^{B'}{}_{C'AA'}..$$

Further, the metricity conditions ${}^{\Gamma}\nabla g_{ab} ={}^{\omega} \nabla g_{ab} = 0$ implies,

$${}^{\Gamma}\nabla_{a}\epsilon_{BC} ={}^{\omega} \nabla_{a}\epsilon_{BC} - \epsilon_{DC}K^{D}{}_{BAA'} - \epsilon_{BD}K^{D}{}_{CAA'},$$

so imposing a symmetry on the contorsion one form $K_{BCAA'} = K_{CBAA'}$, that decomposes then as,

$$K_{ABCC'} = -\frac{1}{2}\sigma_{ABCC'} + 2\epsilon_{C(A}\hat{\Theta}_{B)C'}.$$

Teleparallel Gravitational Lagrangian

The Trautman (*metric-affine*) Einstein-Cartan Lagrangian such that $\eta_{\alpha\beta} = \sqrt{g}\frac{1}{2!}\epsilon_{\alpha\beta\gamma\delta}dx^{\gamma}\wedge dx^{\delta} =: *(dx_{\alpha} \wedge dx_{\beta}) = *\Sigma_{\alpha\beta}$ for co-frame $\theta^{a} = \theta^{a}{}_{\alpha}dx^{\alpha}$ in terms of soldering form $\theta^{AA'}{}_{\mu} = \sigma_{a}{}^{AA'}\theta_{\mu}{}^{a}$ and $\Sigma^{AB} = \frac{1}{2}\theta^{A}{}_{A'} \wedge \theta^{BA'}$ reads,

$$\mathcal{L}_{EC} = -\frac{1}{2}\mathcal{F}_{ab} \wedge \eta^{ab} = \mathcal{F}_{ab} \wedge \ *(\theta^{a} \wedge \theta^{b}). \tag{1}$$

Goldberg's Lagrangian $\mathcal{L}_{CG}$ is the teleparallel version of $\mathcal{L}_{EC}$, formed by appending the Witten-Nester type two-form constructed from the anti-self-dual part of an $so(1,3)$ connection defined (with $\delta^{A}{}_{B} = \epsilon_{B}{}^{A}$) by

$${}^{\Gamma}\bar{\sigma}^{AA'} := i\Gamma^{A}{}_{B}\delta^{A'}{}_{B'} \wedge \theta^{BB'} \tag{2}$$

as a boundary term[5] to the Trautman Lagrangian[6]

$$\begin{aligned}\mathcal{L}_{EC} &= -\frac{1}{2}\mathcal{F}_{ab} \wedge \eta^{ab},\\ \mathcal{L}_{CG} &= \mathcal{L}_{EC} - \frac{i}{2}d(\theta^{a} \wedge \Theta_{a}) + \frac{i}{2}\Theta^{a} \wedge \Theta_{a} - d(\theta_{AA'} \wedge \ {}^{\Gamma}\bar{\sigma}^{AA'}),\\ &= -i\theta^{B}{}_{A'} \wedge \theta^{CA'} \wedge \omega^{A}{}_{C} \wedge \omega_{AB}\end{aligned}$$

[5]The addition of an exact form, boundary term, $d\mu$ to $\mathcal{L}_{EC}$ while affecting the form of the symplectic potential, ϑ does not change the symplectic two-form structure, ϖ: the addition of a total divergence to a Lagrangian does not change the field equations since total divergences have identically vanishing variational derivatives,

$$\hat{\mathcal{L}} = \mathcal{L} + d\mu, \qquad \hat{\vartheta} = \vartheta + \delta\mu, \qquad \hat{\varpi} = \delta\hat{\vartheta} = \delta\vartheta.$$

[6] The 4-form $\frac{i}{2}d(\theta_{a} \wedge \nabla\theta^{a})$ being just the exterior derivative of a translational Chern-Simons

The first variation of which is

$$\begin{aligned}\delta\mathcal{L}_{CG} &= (-i\delta\theta^{B}{}_{A'} \wedge \theta^{CA'} \wedge +i\delta\theta^{CA'} \wedge \theta^{B}{}_{A'}) \wedge \omega^{A}{}_{C} \wedge \omega_{AB} \\ &= -2i\delta\theta^{B}{}_{A'} \wedge \theta^{CA'} \wedge \omega^{A}{}_{C} \wedge \omega_{AB}, \\ &= 2i\psi^{B}{}_{A'} \wedge \theta^{CA'} \wedge \omega^{A}{}_{C} \wedge \omega_{AB}\end{aligned}$$

References

[1] McCrea JD Class. Quantum Gravity,*9*(1992) 553-568.

[2] Mielke EW and Kreimer D,*MZ-TH/97-18.*

[3] Gronwald F, Muench U, Macias A, Hehl FW,*gr-qc/9712063.*

[4] https://eesc.columbia.edu/courses/v1003/lectures/ozone/

[5] Seigel E https://medium.com/starts-with-a-bang/the-bullet-cluster-proves-dark-matter-ex

[6] Mashhoon B https://arxiv.org/pdf/1101.3752.pdf

[7] Plebanski, J.F. (1975). *J. Math. Phys.* **16**, 2395

[8] Hehl FW and Massoon B https://arxiv.org/pdf/0902.0560.pdf

[9] Poincare, Science and Hypothesis, Walter Scott Publishing 1905

3-form,

$$C_T = \theta_a \wedge \Theta^a.$$

The teleparallel theory of gravity is a gauge theory of the translational group with potential, the co-frame in which the curvature of the $sl(2,\mathbb{C})$ connection decomposes according to the teleparallel condition,

$$\mathcal{F}^{A}{}_{B}\delta^{A'}{}_{B'} + \mathcal{F}^{A'}{}_{B'}\delta^{A}{}_{B} = 0,$$

applied to $\mathcal{L}_{EC}$ giving connection and Torsion,

$$\Gamma^{A}{}_{B} = 0,\ {}^{\Gamma}\Theta^{AA'} = d\theta^{AA'}$$

while contorsion is of the form of

$$K^{A}{}_{B} = -\omega^{A}{}_{B}.$$