Symplectic Measures of Contact with Noether

Lee McCulloch-James
Physics Department*, Dwight School, London

April 15, 2020

Abstract

Witten’s covariant symplectic formalism is used to compare a set of cohomologous chiral first-order Lagrangians for gravity through their symplectic structures and thus Liouville/GHS measures that they provide for on-shell trajectories within the phase space of solutions. Their equivalence modulo boundary terms informs the symplectic potentials that are Noether charges. A pedagogical treatment of Noether’s theorem contextualizes the notation of this less familiar covariant phase space picture. To that end also, some background intuition on closely related contact forms within jet bundle framework of field theories is outlined. This is in order to reveal, using a manifestly chiral coupling of a spin $\frac{3}{2}$ field to one of those chiral gravitational Lagrangians, the integrability hierarchy between Weyl, Rarita-Schwinger and Einstein vacuum field equations.

Contents

- Principles of Equivalence and General Covariance .................. 2
- Symplectic Liouville measure ............................................. 3

**Symplectic Calculus of Variations**

- Conserved Noether Currents ............................................. 5
- Geometry of space of Variations ........................................ 7
- Making Contact with Jet Bundles ....................................... 8
- Symplectic two-form and linearised equations ......................... 8
- Diffeomorphism Invariance through Lie Derivative ................. 10

**Cohomologous Lagrangians for Gravity**

- Spinorial Pseudo-Energy-Momentum Charges ......................... 11
- Equivalent Lagrangians and their Super-Potentials .............. 12

**Integrability and Lax Pair for Gravitinos**

- Contact forms for Complex N=1 Supergravity Lagrangian ........ 14
- Half-flat charge from contact form ................................... 15
- Discussion ........................................................................ 16

*https://mcmurmerings.wordpress.com/ leemcjames@gmail.com
Every field in the action is a threat to conservation, and this threat can be met in one of two ways: a given field can have Euler-Lagrange equations or have symmetries. As long as for every field, one condition or the other is satisfied, then there are conservation laws, A. Trautman

An action density involving first order derivatives of the metric (potential) gravitational field is not a scalar density. It would be a scalar if formed straight from the contracted Riemann Curvature tensor, \( \sqrt{-g} F[g^{ab}] \) involving second order derivatives in \( g_{ab} \). Einstein’s introduction of the connection, \( \Gamma^a_{bc} \) as an independent dynamical variable comes at the loss of this invariance, [1]. As a result, the conservation law arising from such a Palatini-like density has a current representing the energy-momentum of the gravitational field that is not a tensor, merely a pseudo-tensor. [1]

Principles of Equivalence and General Covariance

According to the Einstein’s *Equivalence Principle*, inertial mass can be distinguished from gravitational mass only non-locally. Being indistinguishable at a point makes for unhappy marriage with the *Principle of General Covariance*. This is partially resolved by saying that physical energy-momentum is quasi-local, rather defined within closed 2-surfaces. That a formulation of gravitational energy cannot be localised means it is not possible to obtain an expression for the energy of the gravitational field satisfying both of the following conditions:

I. when added to other forms of energy the total energy is conserved,

II. the energy within a definite three volume region at a certain time is independent of the co-ordinate system.

All that a pseudo-tensor satisfies is the first case. In practical modelling situations neither conditions apply but we can deliver suitable field equations using a Palatini-style Einstein-Cartan Lagrangian, [2][3]

\[
\mathcal{L}_{EC} = -\frac{1}{2} \mathcal{F}_{ab} \wedge \eta^{ab} = \mathcal{F}_{ab} \wedge *(\theta^a \wedge \theta^b),
\]

(1)

where \( \eta^{ab} = \sqrt{-g} \epsilon^{ab}_{\gamma\delta} dx^\gamma \wedge dx^\delta =: *(dx^a \wedge dx^b) \) for co-frame \( \theta^a = \theta^a_{\alpha} dx^\alpha \) and soldering form, \( \theta^a_{\mu} \) that marries free falling 4-d Lorenzian indices, \( a \) to spatial ones, \( \mu = 0, 1, 2, 3 \).

---

1. The Hilbert or Moeller’s tetrad action that is quasi-invariant under an (infinitesimal) coordinate transformations (the action of the Lie derivative w.r.t a vector field \( \xi^\mu \) that is timelike, and in which there is a coordinate system such that \( \xi^\mu \)’s components are (1,0,0,0).)
2. Trautman *metric-affine*
Symplectic Liouville measure

The addition of a total divergence to a Lagrangian does not change the resulting field equations since total divergences have identically vanishing variational derivatives. This freedom to have different boundary terms, \( B \), both determines and fixes the quasi-local energy-momentum and boundary conditions. The (pre)symplectic form \( \varpi \) is defined \([5],[6]\) as the functional exterior derivative, \( \delta \) of the Lagrangian’s symplectic potential, \( \vartheta \),

\[
\varpi(\phi, \delta_\lambda \phi, \delta_\epsilon \phi) = \delta_\epsilon \vartheta(\phi, \delta_\lambda \phi) - \delta_\lambda \vartheta(\phi, \delta_\epsilon \phi).
\] (2)

Here \( \delta_\epsilon \phi(x) \) is viewed as a one-form, \( \delta_\epsilon \phi(x) \in \Lambda^1(\mathcal{E}) \) on the submanifold, \( \mathcal{E} \subset \Gamma \) of solutions of the equations of motion, \( E_A = 0 \) within the space of all histories, \( \Gamma \). Adding an exact form, boundary term, \( B = d\mu \) to \( \mathcal{L}_{EC} \), while affecting the form of the symplectic potential, \( \vartheta \) does not change the symplectic two-form structure, \( \varpi \),

\[
\hat{\mathcal{L}} = \mathcal{L} + d\mu, \quad \hat{\vartheta} = \vartheta + \delta \mu, \quad \hat{\varpi} = \delta \hat{\vartheta} = \delta \vartheta.
\]

That is, since \( \delta^2 \mu = 0 \), the symplectic two form, \( \varpi \) remains invariant under the addition of a boundary term locally constructed from generic field variables, \( \mu = \mu(\phi^A) \) to a Lagrangian. Defining, in addition to the wedge, \( \wedge \)-product acting on \( r \)-dimensional spatial forms \( \delta \phi^A \in \Lambda^r(M) \), a \( \vee \)-Grassman product acting on the forms of the solution space, \( \delta \phi^A \in \Lambda^k(\mathcal{E}) \), so

\[
\delta \phi^A(x) \vee \delta \phi^A(y) = -\delta \phi^A(y) \vee \delta \phi^A(x)
\] (3)

we have that the symplectic form and its potential commute under a double graded commutative product,

\[
\varpi \wedge \vee \vartheta \equiv \varpi \vee \vartheta = (-1)^{rs+k+1} \varpi \vee \vartheta = (-1)^{2+r+s+1} \varpi \vee \vartheta.
\] (4)

While Noether’s symmetries theorems do not care about the Equivalence Principle - delivering currents in any coordinate system - the presence of an additional boundary term will necessarily change the value of the Lagrangian’s conserved charges. For example, consider the Lagrangian of the classical field theory for Electro-Magnetism, \( \mathcal{L} = -\frac{1}{4} F_{ab}F^{ab} \) whose four component 4-vector potential \( A_\mu \) are the independent dynamical fields built into the correlated components of the Faraday field, \( F = dA = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \). With source current, \( J \) we have

\[
\mathcal{L}_M = \frac{1}{2} dA \wedge^* dA + AJ, \\
\vartheta_M = \int_\Sigma \delta A \wedge^* F \in \Lambda^1_E(M), \\
\varpi_M = \int_\Sigma \delta^2 A \vee \delta^* F - \delta A \vee \delta^* F \in \Lambda^2_4 E, M.
\] (5)

Geometrically we will view the Hamiltonian 3-form as associated to a first

---

3It is only by specifying conditions at the boundary that the field equations of motion reveal their unique deterministic world-line trajectories.

4in the cotangent space to \( M \)

5By the gauge indistinguishability principle, that the boundary terms, \( B \) are not arbitrary means they must have a physical significance.

6Through the Legendre transformation \( L = p\dot{q} - H \) equivalent for fields \( \phi \) with conjugate momentum \( \pi \)
order Lagrangian 4-form by contracting the Lagrangian with a timelike space-time displacement vector field, $\xi$ that specifies the time evolution:

$$\xi \mathcal{L} = \mathcal{L}_\xi \phi \wedge \pi - H(\xi),$$

with the Lie derivative being related to the exterior derivative, $d$ and the interior product, $\lrcorner$ acting on a k-form, $\varphi$ according to,

$$\mathcal{L}_\xi \varphi = d\xi \lrcorner \varphi + \xi \lrcorner d\varphi.$$  

such that $\xi \lrcorner H = H(\xi) \equiv H_\xi$ is viewed as the generator of time evolution - a result of there being no time derivatives of $g_{00}$ in the Lagrangian, with energy-momentum charge current, $\sigma$ implicitly defined within a 2-surface, $\partial \Sigma$ through the vanishing of the translational Noether 3-current

$$H_\xi = \mathcal{L}_\xi \phi \wedge \pi - \xi \lrcorner \mathcal{L} \equiv \xi^\mu \mathcal{H}_\mu + d\sigma(\xi) = \int_{\partial \Sigma} \sigma(\bar{\phi}, \xi)$$

for on-shell field solutions, $\bar{\phi}$. Liouville’s theorem is a statement that under this evolution both the symplectic form and the Liouville measure, $\varsigma$ of phase space are invariant,

$$\mathcal{L}_\xi \varsigma - 0 \implies \mathcal{L}_\xi \varsigma = 0.$$  

This natural measure as picked out by the dynamics on $2n$-dimensional phase space is related to the symplectic form according to,

$$\varsigma = \frac{(-1)^n (n-1)/2}{n!} \bar{\omega}^n$$

The space of of physical trajectories obeying the Hamiltonian constraint, $\mathcal{E} \subset \Gamma$ has 2 dimensions lower than the full phase space. Gibbons et al (GHS), construct a measure induced from this on the space of trajectories, $\mathcal{E}$

$$\varsigma_{\mathcal{E}} = \frac{(-1)^{(n-1)(n-2)/2}}{(n-1)!} \bar{\omega}^{(n-1)}, \text{ where } \bar{\omega} = \bar{\omega} + d\mathcal{H}_\xi \wedge d\xi$$

Carrol has argued, for the use of this measure in considerations of the smooth state of the early universe in which its microstate is randomly drawn from this very measure on the space of Cosmological trajectories. This can then be used to quantify how much fine-tuning underpins the conventional assumption of a smooth universe near the Big Bang to explain apparent equilibrium on fine scales baked into the Cosmic Microwave Background.

With this and possible alternative formulations of gravity in mind, we will use the manifestly covariant dynamical description to compare a set of cohomologous chiral Lagrangians for gravity and their symplectic structure, $\varsigma$ on the

\[ \int_{\Sigma} \varsigma = \int_{\Sigma} \varsigma \]

\[ \int_{\Sigma} = \int_{\Sigma} \varsigma \]
phase space of solutions to their equations of motion, $\mathcal{E}$. The pre-symplectic structures of these first order Lagrangians for Einstein’s field equations are collated and their equivalence modulo boundary terms is made manifest.

A pedagogical treatment of Noether’s theorem is also presented with a view of contextualizing the notation of the less familiar covariant phase space picture. To that end also, some background intuition on the closely related jet bundle picture of contact forms is also provided, with a manifestly chiral coupling, [10], [11] of a spin $\frac{3}{2}$ field to the first order Lagrangians providing a concrete example of the use of these contact structures in order ultimately to explain the integrability hierarchy between Weyl, Rarita-Schwinger and Einstein vacuum field equations.

Symplectic Calculus of Variations

Taking the Bergmann approach to observables, [9] generally covariant theories in phase space have in common that the Hamiltonian is a linear combination of first class constraints. This means that the Hamiltonian vanishes “on shell”, i.e., when the equations of motion are satisfied. Certain combinations of first class constraints thus generate gauge symmetries, And since rigid translation in time coordinate is a spacetime diffeomorphism which does engender corresponding gauge symmetries of dynamical variables in configuration-velocity space, some authors have concluded that the Hamiltonian is itself a symmetry generator.

The Noether current can be expressed as consisting of a term which vanishes when the field equations are satisfied and a term whose divergence vanishes identically. Adherence to Noether follows from the vanishing of the divergence of two contributions. We note that in generally covariant theories the energy conservation law can be re-written, using the Euler–Lagrange equations, such that it holds “identically.” The boundary term does not interfere with the existence of a conserved charge, and that for many important examples it is non-zero and indeed contributes to the charge.

Conserved Noether Currents

As Trautman argues every field, $\phi(x)$ in the action you can construct is a threat to a potential Conservation law. This threat can be met in one of two ways: have that field be an a posteriori dynamical solution to the resulting Euler-Lagrange equations or insist a priori it be kinematical generator of symmetries. As long as for every field, one condition or the other is satisfied, there are conservation laws. You can thus ask different things of your field variable in a Variational Principle. We follow closely the pedagogical text of Banados et al, [12] in this review of the process. Readers familiar with the process need only skim to pick up the the notational conventions used subsequently. The two types of complimentary variation are:

Symmetry, $\delta_s \phi(x)$ these will always be deformations of the fields, not of the coordinates. Their variations $\delta_s \phi(x)$ are constrained a priori to satisfy an equation, while the “fields” $\phi(x)$ are totally arbitrary.
On-shell, $\delta \tilde{\phi}(x)$ these variations are opposite to a symmetry such that the fields $\phi(x)$ are constrained \textit{a posteriori} to satisfy their Euler-Lagrange equations [as $\phi(x) = \tilde{\phi}(x)$ sits within a subset of the solution space of all possible variational paths], while the variations $\delta \tilde{\phi}(x)$ are arbitrary.

In turn then, of \textbf{symmetry variations} we need only define a symmetry, in the weak sense as a function, $\delta_s \phi(x)$ such that for any $\phi(x)$ the action, $S$ say

$$S[\phi(x)] = \int d^4x L(\phi, \partial_{\mu} \phi),$$  \hspace{1cm} (13)

is invariant,

$$\delta S[\phi, \delta_s \phi] \equiv S[\phi + \delta_s \phi] - S[\phi] = \int d^4x \partial_{\mu} B^\mu,$$  \hspace{1cm} (14)

up to a boundary term, $B$. This boundary term, $B$ does not preclude the existence of a charge but rather contributes to it. We take note of the notation, $S(\phi(x))$ as a functional of $\phi(x)$ the coordinates, $x$ are summed over as dummy variables. So that the definition of symmetry, does not involve changes in the coordinates: even symmetries associated to spacetime translations, rotations, can always be written as some transformation $\phi(x)$ acting on the field. $\delta S(\phi, \delta_s \phi)$ denotes the variation of the action under the symmetry making explicit the action’s dependency on both field configuration and its symmetry variation. To be sure note that given any function, $f(x)$ its variation, $f(x + dx) - f(x) =: f'(x) dx \equiv df(x, dx)$ depends on both the point, $x$ and the magnitude for the variation $dx$. The set of symmetries of a field theory action is defined as that set of all infinitesimal functions $\delta_s \phi(x)$ for arbitrary, $x$ that vary the action up to some divergence of a boundary term \textit{symplectic potential}, $\vartheta$. \hspace{1cm} (14) is an equation for the function $\delta_s \phi$ not for $\phi$ as $\delta_s \phi$ is a symmetry only if it holds for all $\phi$.

For \textbf{on-shell variations} consider actions of the \textit{first order} form \hspace{0.2cm} (13) dependent on field and first order derivative with consequent Euler-Lagrange equation, $E$

$$E[\phi(x)] \equiv \partial_{\mu} \frac{\partial L}{\partial \phi_{,\mu}} - \frac{\partial L}{\partial \phi} = 0.$$  \hspace{1cm} (13)

The on-shell variation is computed as

$$\delta S[\phi, \delta \phi] = \int d^4x \left( \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial \phi_{,\mu}} \delta \phi_{,\mu} \right),$$

$$= \int d^4x E[\phi(x)] \delta \phi + \partial_{\mu} \left( \frac{\partial L}{\partial \phi_{,\mu}} \delta \phi \right) = \int d^4x \partial_{\mu} \left( \frac{\partial L}{\partial \phi_{,\mu}} \delta \phi \right).$$  \hspace{1cm} (15)

where we have used that the field $\tilde{\phi}$ satisfies $E[\phi] = 0$. While we have that \hspace{0.2cm} (14) is valid for any $\phi$ and in particular for the on-shell solution, $\tilde{\phi}$ \hspace{0.2cm} (15) is valid for any $\delta \phi$ in particular $\delta_s \phi$. Putting these particular versions into \hspace{0.2cm} (14) and \hspace{0.2cm} (15) so that their left hand sides are equal and subtracting, we obtain the conserved current equation that is Noether theorem,

$$\partial_{\mu} j^\mu = 0 \quad \text{where} \quad j^\mu \equiv \frac{\partial L}{\partial \phi_{,\mu}} \delta \phi(x) - B^\mu.$$  \hspace{1cm} (16)
Geometry of space of Variations

The cotangent space, $T^*M$ consists thus of all equivalent first-order approximations in a Taylor approximation sense of the function, its elements being linear functionals on $TM$. It is therefore the space of normed functionals of linear maps, $\delta \phi$.

$TM$ tangent space the set of derivatives, linear approximations of a function, *linear form*, from $M \rightarrow \mathbb{R}$. $TM$ is set of all possible velocity vectors, of parameterised curves on $M$. If you fix a point on $M$, then the set of all possible velocity vectors there has a natural vector space structure.

$T^*M$ cotangent space the set of derivatives from $\mathbb{R} \rightarrow M$. The set of all possible derivatives of real-valued functions defined by composing the function with a parameterised curve. The value of the derivative at the point depends only on the velocity vector of the curve and is a linear function of it and the set of all possible derivatives of the function is dual to $TM$.

Accordingly we consider, $\phi$ as a one parameter family of field configurations, with some possible internal and tangent space indices, $A$ suppressed. A variation of some local function $f(\phi(\lambda))$ (of a $\lambda$ parameterised curve, at a given space-time point) is denoted by

$$\delta \lambda f = \frac{d}{d\lambda} f(\phi(\lambda)).$$

The (first order) action functional, $S$ is viewed as a scalar function on the space of all field variables and their first derivatives. The variation of a field $\delta \phi$ is a tangent vector to this space and the first variation of the action, $\delta S$ is viewed as an exact form (functional of $\delta \phi$) on the space of all histories, $\Gamma$,

$$\delta S = \int_M \delta L(\phi, \nabla \phi) =: dS(\delta \phi).$$

(17)

Explicitly the first variation as an exact form with possible mixed indices $A$ made explicit reads,

$$dS(\delta \phi^A) = \int_M E \wedge \delta \phi^A + \int_{\partial M} j^\mu(\delta \phi^A)\eta_\mu.$$  

(18)

Here $\eta_\mu$ is a three volume form basis and $E_A \wedge \delta \phi^A = \frac{\delta L}{\delta \phi^A}\delta \phi^A$ is the functional derivative four form which vanish when $\delta \phi^A = \delta \phi^A$ is a solution to $E_A = 0$. We then define the *presymplectic potential* $\vartheta_\Sigma$ as a 1 form, defined in terms of the canonical momentum, $\pi$ or Noether current $j^\mu$ as,

$$\vartheta_\Sigma(\delta \phi) = \int_\Sigma \delta \phi \wedge \pi = \int_\Sigma j^\mu(\delta \phi)\eta_\mu,$$  

(19)

so that (suppressing indices) [18] reads

$$dS = \int_\Sigma \delta L = \int_\Sigma E \wedge \delta \phi + d\vartheta,$$  

(20)

for some Cauchy surface, $\Sigma$ in $M$ in which we are *on-shell* so that the equations of motion, hold $E_A = 0$ in the submanifold, $E \subset \Gamma$ of solutions to the equations of motion.
Making Contact with Jet Bundles

General Relativity, treated as a gauge theory of the Lorentz group can be formulated using the soldering form (co-frame dual to the tetrad field) from Cartan’s multi-vector Calculus. A so-called Cartan G-structures involves the soldering of the cotangent bundle, $T^*B$ to spacetime, $M$. A brief heuristic overview follows of the cotangent space on which the symplectic analysis will take place later.

The configuration space for $\phi^A$ is the fibre bundle $\pi: B \to M$. The field theoretic analogue of the tangent bundle of particle mechanics is the Jet Bundle over this fibre bundle, $J^n(B, M)$. The action, $S$ is a function on this bundle. So for instance, the Einstein-Hilbert action $S_H[J^2g]$ is second order in the metric $g$ (where $J^2g = g, \partial g, \partial^2 g$) while metric-affine Palatini-like Lagrangians are first order in the dynamical connection, $\Gamma$, $S_{Pal}[J^1g, J^1\Gamma]$ in which $\Gamma$ is determined from $g$ algebraically.

Within the symplectic geometry of phase space, the cotangent bundle $T^*B$ does not need an (extra) metric structure on the basic world sheet, $M$ to define the differential of a function. However you do need a metric in order to define the gradient on its dual tangent bundle, $T^B$. At a point, both $T^B$ and $T^*B$ are real vector spaces of the same dimension and therefore possess many isomorphisms to each other. The introduction of a Riemannian metric, $g$ (resp. symplectic form, $\epsilon$) gives rise to a natural isomorphism between the tangent and cotangent space, associating to any tangent co-vector a canonical tangent vector. The tangent co-vector of $T^*B$ is called the canonical one-form, or the contact form $\kappa$ on the dual space to the jet bundle of field configurations, $\kappa \in \Lambda^1[J^1(B, M)^*]$. In the multi-symplectic covariant approach previously discussed, the symplectic potential, $\vartheta$ would be the canonical $(n-1)$-form on the covariant cotangent bundle, $T^*B$.

Symplectic two-form and linearised equations

The (pre)symplectic 2 form $\varpi$ is defined as the functional exterior derivative, $\delta$ of the potential $\vartheta$,

$$\varpi(\phi, \delta \lambda \phi, \delta \epsilon \phi) = \delta \epsilon \vartheta(\phi, \delta \lambda \phi) - \delta \lambda \vartheta(\phi, \delta \epsilon \phi).$$

(21)

This two form, $\varpi$ being additive and anti-symmetric in its dependence on each of the perturbed fields means that the bilinear product,

$$\varpi(\phi^A, \delta \lambda \phi^A, \delta \epsilon \phi^B) = \varpi_{AB} \delta \lambda \phi^A \delta \epsilon \phi^B$$

is necessarily antisymmetric in $[A, B]$. Accordingly the linearised solutions of $\phi^A$, $\dot{\phi}$ can be regarded as anti-commuting (or Grassman-valued) one forms $\dot{\phi}(x)$ on the tangent space to the symplectic space. We can drop the subscript parameters (as implicit) and rewrite the form compactly as,

$$\varpi(\delta \dot{\phi}^A) = \delta \vartheta(\delta \dot{\phi}^A) \quad \text{or} \quad \varpi(\dot{\phi}^A) = \delta \vartheta(\dot{\phi}^A).$$

\footnote{An important bundle, $B$ is the spin (vector) bundle with fibres $C^2$ associated to the principal $SL(2, C)$ fibre bundle (PFB) that double covers the PFB of space-time oriented, orthonormal frames whose fiber is diffeomorphic to the proper Lorentz group}

\footnote{$A$ here represents some tangent and/or internal index}
We have that \( \delta \varepsilon \phi(x) \) is viewed as a one form on the submanifold, \( \mathcal{E} \) of solutions of the equations of motion, \( E_A = 0 \) in the space of all histories, \( \Gamma \), \( \mathcal{E} \subset \Gamma \):

\[
\delta \phi^A(x) \vee \delta \phi^A(y) = -\delta \phi^A(y) \vee \delta \phi^A(x) \tag{22}
\]

where \( \vee \) is the grassman product on the solution space and the functional exterior derivative, \( \delta \) maps \( k \)-forms \( \alpha \in \Lambda^k(\mathcal{E}) \) to \( k+1 \) forms such that for \( \beta \in \Lambda^l(\mathcal{E}) \),

\[
\delta^2 \alpha = 0, \\
\delta(\alpha \vee \beta) = \delta \alpha \vee \beta + (-)^l \alpha \vee \delta \beta. \tag{23}
\]

With two degree types on these forms one could define a double graded commutative product, \( \diamond \) such that for \( \alpha \in \Lambda^k(\mathcal{E}) \), \( \beta \in \Lambda^l(\mathcal{E}) \) we have

\[
\alpha \wedge \vee \beta \equiv \alpha \diamond \beta = (-1)^{rs+kl} \beta \diamond \alpha. \tag{24}
\]

Because mixed parameterisations, commute (\( '[\delta \epsilon, \delta \lambda] = 0' \)) we have that a second variation of the action \( \delta^2 S \) realises,

\[
d\varpi = \delta^2 S = \int_{\Sigma} (\delta \lambda E_A) \delta \epsilon \phi^A - (\delta \epsilon E_A) \delta \lambda \phi^A = 0, \tag{25}
\]

for fields \( \dot{\phi}^A \) satisfying the linearised equation of motion, \( \delta E_A \equiv \dot{E}_A = 0 \). Crucially the fields \( \dot{\phi}^A \) need not be a solution to the Euler-Lagrange equation of \( \delta S \) for this closure condition of the symplectic form to hold. Consider \( \delta S \) itself an exact form on space of all histories,

\[
\delta S(\dot{\phi}^A) = \int_M \left( \frac{\delta L}{\delta \dot{\phi}^A} \dot{\phi}^A + dj(x) \right) =: \dot{\phi}^A \cdot dS, \\
= \int_M dj(x) = \int_{\partial M} j(x) \quad \text{for} \quad E_A(\dot{\phi}^A) = 0, \quad j(x) := \frac{\delta L}{\delta \dot{\phi}^A} \dot{\phi}^A(x). \tag{26}
\]

So for a given Cauchy surface \( \Sigma \) our symplectic potential is,

\[
\varpi(\dot{\phi}^A) = \dot{\phi}^A \cdot \varpi := \int_{\Sigma} j(x). \]

We have because of \( E_A \delta \dot{\phi}^A = \frac{\delta E_A}{\delta \dot{\phi}^A} \delta \dot{\phi}^A, \delta \dot{\phi}^A = 0 \) and also because of\( \boxed{25} \) that

\[
\delta^2 S = \int_M \frac{\delta L}{\delta \phi^A} \delta \phi^A + \delta dj(x) = d\varpi, \\
= \int_M \delta E_A \dot{\phi}^A + E_A \delta \dot{\phi}^A + \delta dj(x) = 0, \\
\text{so}, \quad \varpi = \int_{\partial M} \delta j(x) = \int_{\Sigma} \delta j(x) = 0, \quad \text{with linear eqn satisfied,} \quad \delta E_A = \dot{E}_A = 0, \tag{27}
\]

which as it is independent of the Cauchy surface means the flux of symplectic current is conserved across such surfaces. For fields satisfying the linearised equations \( \boxed{14} \) of motion \( \delta E = \dot{E} = 0 \) the symplectic form is closed, \( d\varpi = 0 \), although the fields, \( \phi \) need not be a solution of the equation of motion \( E(\phi) = 0 \) for this closure condition to hold.
Diffeomorphism Invariance through Lie Derivative

In this section we will elaborate the proceeding formalism to illustrate the Diffeomorphism invariance and symmetry aspects for Gauge theories and General Relativity in particular by linking its energy-momentum pseudotensors to Nester’s\[15] quasilocal quantities.

Our geometric variational picture, \[16\] is the following. For canonical momentum field, \(\pi\) defined in \[19\], where we denote our Euler-Lagrange expression as \(E = \delta\bar{\phi} L \equiv \frac{\delta L}{\delta \phi}\), we implicitly define the two variational derivatives \(\delta L\) of the Lagrangian,

\[
L = d\phi \wedge \pi - H(\phi, \pi),
\]

according to,

\[
\delta L = d(\delta \bar{\phi} \wedge \pi) + \bar{\delta} \delta \bar{\phi} L + \delta L \wedge \bar{\delta} \pi.
\]

The natural interpretation of a symmetry is a transformation acting on the fields, \(\phi\), not the co-ordinates, \(x\). The latter rather are dummy variables that can be changed without affecting the value of the integral. The Lagrangian is a scalar since its form may change when the coordinates are changed, but not its value. Within the Lagrangian, all symmetries - even those symmetries whose origin are variations of the coordinates- can be expressed as transformations acting on the fields. So even a symmetry which could be interpreted as variations of coordinates such as spacetime translations \(x^\mu \rightarrow x'^\mu = x^\mu + \epsilon^\mu\) is to be understood as a transformation of the field. That is, such that given a \(\phi(x)\) we build a new translated \(\phi'(x)\) so that the variation of the field is associated to a translation of coordinates,

\[
\phi'(x) = \phi(x - \epsilon) \simeq \phi(x) - \epsilon^\mu \partial_\mu \phi(x),
\]

\[
\delta \phi(x) = -\epsilon^\mu \partial_\mu \phi(x),
\]

and is a local relation involving only the single point \(x\) (and not \(x'\)). For Noether’s theorem we need the infinitesimal version of equations expressing the variations of components of various fields when a given transformation of the coordinates is applied. For the scalar \(\phi'(x + \xi) = \phi(x)\) and a one form \(A'(x + \xi) = \phi(x)\) to linear order in \(\xi\) we have:

\[
\delta \phi(x) = \phi'(x) - \phi(x) = -\xi^\mu \partial_\mu \phi(x) =: -\mathcal{L}_\xi \phi(x),
\]

\[
\delta A_\mu(x) = A'_\mu(x) - A_\mu(x) = -\xi^\alpha \partial_\alpha A_\mu - A_\alpha \partial_\mu \xi^\alpha =: -\mathcal{L}_\xi A_\mu(x).
\]

These are the definition of the \textit{Lie derivative} along a vector \(\xi^\mu(x)\) as an operation acting on tensors that makes no reference to a connection. \^2 Diffeomorphism invariance\[^3\] as generated by \(\xi^\mu(x)\) is realised by the Lie-derivative acting on \(28\) with respect to that field such that \(\delta \rightarrow \mathcal{L}_\xi\) in \(29\) so we have

\[
d\xi_\mu \mathcal{L} \equiv \mathcal{L}_\xi \mathcal{L} = d(\mathcal{L}_\xi \phi \wedge \pi) + \mathcal{L}_\xi \bar{\phi} \wedge \delta \bar{\phi} \mathcal{L} + \delta \pi \mathcal{L} \wedge \mathcal{L}_\xi \pi
\]

An example of the two-form boundary charge term, \(\sigma\) will be given in \[40\] being the time evolution generator,

\[
\mathcal{H} = \int_\Sigma \xi^\mu \mathcal{H}_\mu + d\sigma(\xi) = \int_{\partial \Sigma} \sigma(\xi).
\]
Cohomologous Lagrangians for Gravity

In Witten’s covariant symplectic formulation, the symplectic structure is the functional derivative of the boundary term (symplectic potential) that results from performing the ‘integration by parts’ in the Variational Principle. Consider again the Einstein-Cartan Lagrangian, decomposed into its complex (anti)chiral Trautman forms. Defining the complex chiral two form of Plebanski, \( \Sigma^{AB} = \frac{1}{2} \theta^{A}_{\ A'} \wedge \theta^{B}_{\ B'} \) and the Curvature of a general affine connection, \( \Gamma \) as \( F^{ab} \equiv \Gamma^{d}_{\ ab} \), we have,

\[
\mathcal{L}_{EC} = -\frac{1}{2} F^{ab} \wedge \eta^{ab} = -\frac{1}{2} [F^{+} (\Gamma^{+}) + F^{-} (\Gamma^{-})] \wedge \eta^{ab},
\]

\[
= L_{EC}^{+} + L_{EC}^{-} = -\frac{i}{2} [F^{ab}_{\ +} \wedge \Sigma^{ab} - F^{ab}_{\ -} \wedge \Sigma^{ab}],
\]

\[
\leftrightarrow -i [F_{A'B'}^{ab} \wedge \Sigma^{A'B'} - F_{AB}^{ab} \wedge \Sigma^{AB}].
\]

The symplectic potential for \( \mathcal{L}_{EC} \) will be of the form \( \vartheta = \delta \Gamma_{ab} \wedge \eta^{ab} \). In order to make the Nieh-Yan form of the generating functional more apparent we will use a complex form of the Witten-Nester, two form, \( \Gamma_{\bar{\sigma}}^{AA'} = \Gamma^{\Theta AA'} - \bar{D} \theta^{AA'} \), where \( \bar{D} \) is the covariant derivative associated to the self-dual connection \( \bar{\Gamma}^{\ A\ B} \).

These boundary terms will induce some change in the physical interpretation of the theory, modifying the notion of energy and choice of boundary conditions.

Spinorial Pseudo-Energy-Momentum Charges

Einstein’s field equations, for a symmetric, metric compatible connection, \( \omega^{ab} \), read,

\[
\omega \nabla (\theta^{A\ A'} \wedge \theta^{B\ A'}) = 0.
\]

Upon pulling out an exact form and using Cartan’s first structural equation this reads

\[
-i d^{\omega} \bar{\sigma}^{AA'} = -4 \pi i \Xi^{AA'},
\]

where \( \Xi^{AA'} := T^{AA'} + t^{AA'} \) is the total (matter+field) energy-momentum 3-form for Sparling 3-form, \( t^{AA'} \) given by

\[
t^{AA'} := \frac{i}{4 \pi} (\omega^{A\ B}_{\ \ B'} \wedge \omega^{B\ B'}_{\ B'}/ \Theta^{AA'}).
\]

This gravitational field energy-momentum has the components of a pseudo-tensor and is exact only for Torsion, \( \Theta^{AA'} \)-free connections that are Ricci flat. That is, \( dt^{AA'} = 0 \) holds for vacuum Einstein only. Such a construction is peculiar to the torsion-free connection case because of the first Bianchi identity,

\[
\omega \nabla \Theta^{AA'} = \Omega^{A\ B}_{\ B'} \wedge \theta^{B\ A'} + \Omega^{A'\ B'}_{\ B} \wedge \theta^{A\ B'} = 0.
\]

\[\text{11} \text{For symmetric metric connection we have connection and Torsion, } \Gamma^{A\ B} = 0, \Gamma^{\Theta AA'} = d\theta^{AA'}, \text{ so contorsion is of the form } K^{A\ B} = -\omega^{A\ B}.\]
Equivalent Lagrangians and their Super-Potentials

The equivalence modulo divergences of the metric Lagrangians implies equivalence modulo expressions for quasilocal conserved charges and choice of appropriate boundary conditions. Accordingly we collect, \cite{19,17} here various complex, \cite{20} Chiral Lagrangians, respectively of Capovilla et al., \cite{21} \cite{22} \cite{23}, \cite{24} \cite{25} Nieh-Yan, \cite{26}, and Goldberg,\cite{27}

\[
\mathcal{L}_{SSJ} = 2\mathcal{L}_{EC} = i\theta^A_{\alpha'} \wedge \theta^{B\alpha'} \wedge \mathcal{F}_{AB}(\Gamma)
\]
\[
\mathcal{L}_{NY} = \mathcal{L}_{SSJ} - \frac{i}{2} \Theta^{A\alpha'} \wedge \Theta_{A\alpha'}
\]
\[
\mathcal{L}_{g^3_*} = iD\theta^{A\alpha'} \wedge D\theta_{A\alpha'},
\]
\[
\mathcal{L}_{CG} = -i\theta^A_{\alpha'} \wedge \theta^{C\alpha'} \wedge \omega^A_C \wedge \omega_{AB}.
\]

To make the link to the multisymplectic formalism we will tie these together by looking at their relationships to each other. We see for instance that \(\mathcal{L}_{NY}\) can be derived from \(37\) by adding to it the exact four form \(\frac{i}{2}d(\theta_a \wedge D\theta^a)\). This 4-form being just the exterior derivative of a translational\cite{22} Chern-Simons 3-form,

\[
C_T = \theta_a \wedge \Theta^a.
\]

There are though two Chern-Simon terms, one for each dynamical variable of our metric-affine gauge theory. They yield boundary terms \(dC\) which are generating functions which in the case of a Ashtekar’s Hamiltonian reformulation of GR \cite{16} induces a new pair of canonical variables. We have in a Riemann–Cartan geometry, the Chern–Simons term \cite{28} for the Lorentz connection, \(C_L\)

\[
C_L = \Gamma^a_b \wedge \mathcal{F}^b_a - \frac{1}{3} \Gamma^a_b \wedge \Gamma^b_c \wedge \Gamma^c_a
\]

together with this translational Chern–Simons term \(C_T\) give rise to boundary terms, \(dC\) whose variational derivative with respect to the connection and co-frame generate respectively the first and second Bianchi identities. The Lagrangians\cite{13} are related to each other according to:

\[
\mathcal{L}_{EC} = -\frac{1}{2} \mathcal{F}_{ab} \wedge \eta^{ab},
\]
\[
\mathcal{L}_{SSJ} = \mathcal{L}_{EC} - \frac{i}{2} d(\theta^a \wedge \Theta_a) + \frac{i}{2} \Theta^a \wedge \Theta_a,
\]
\[
\mathcal{L}_{g^3_*} = \mathcal{L}_{SSJ} + id(\theta^{A\alpha'} \wedge D\theta_{A\alpha'}),
\]
\[
\mathcal{L}_{CG} = \mathcal{L}_{SSJ} - d(\theta_{A\alpha'} \wedge \omega^A \bar{\sigma}_{A\alpha'}).
\]

The respective symplectic 3-form potentials\cite{14} read

\[
\vartheta_{EC} = -\delta\omega_{ab} \wedge \eta^{ab},
\]
\[
\vartheta_{SSJ} = i\delta\omega_{AB} \wedge \theta^{A\alpha'} \wedge \theta^{B\alpha'},
\]
\[
\vartheta_{g^3_*} = \vartheta_{SSJ} + i\delta\theta_{A\alpha'} \wedge D\theta^{A\alpha'},
\]
\[
\vartheta_{CG} = \vartheta_{SSJ} - \delta\theta_{A\alpha'} \wedge \Gamma^A \bar{\sigma}_{A\alpha'}.
\]

\footnote{The teleparallel theory of gravity is a gauge theory of the translational group with potential, the co-frame.}

\footnote{That various metric connection theories possess distinct boundary terms could in principle be revealed by considering pulsar-white dwarf binary systems used as test cases for exploring the continued wider validity of the General Theory of Relativity, GR over some its competitor theories.}
The superpotential and Witten-Nester type two-form constructed from the anti-self-dual part of an so(1, 3) connection are defined (with $\delta^A_B = \epsilon_B^A$) as
\[
\omega^\sigma_{AA'} := i\omega^A_B \delta^{A'}_{B'} \theta^{BB'}, \quad (39)
\]
\[
\Gamma_{\sigma}^{AA'} := i\Gamma^A_B \delta^{A'}_{B'} \wedge \theta^{BB'} \quad (40)
\]
Recalling \[20\]
\[
dS = \int E \wedge \delta\phi + d\vartheta,
\]
for some Cauchy surface, $\Sigma$ in $M$ we see that on-shell each of the different symplectic potentials have distinct canonical momenta realising distinct Noether currents $j^\mu$ as defined in \[19\]
\[
\vartheta(\delta\phi) = \int \delta\phi \wedge \pi = \int j^\mu(\delta\phi)\eta_a.
\]
The Noether charges will arise from variations of both the metric (co-frame) and affine (connection) variables of the theory and will need different surface charge densities fixed on the boundary. The Hamiltonians $H$ derived from these cohomologous Lagrangian scalar densities \[5\] will respectively have variations with different boundary terms that vanishes either by fixing the potential as (per Dirichlet), the connection, $\omega$, or both connection and co-frame, $\theta$ as in the case of the teleparallelism formalism, \[27\] $L_{CG}$. Their respective quasi local charge definitions derived by differing boundary terms will possess distinct physical interpretations.\[6\]
Integrability and Lax Pair for Gravitinos

We will look now to apply some of the proceeding jet-bundle contact form formalism, [13]. The Rarita-Schwinger Lagrangian, $L_3^2$ has associated Noether symmetries conditioned on their Integrability according to a variation delivering Einstein Vacuum equations. We will note, that the gauge freedom in the contact form structure associated to $L_3^2$ has a Twistor, [30] [31] [32] interpretation, written as it can be, in terms of coupled (Lax-paired) Weyl equations of the potentials.

Contact forms for Complex N=1 Supergravity Lagrangian

Consider the chiral coupling program summarised by Pillin, [10], as implemented Jacobson et al, [24],[23] [25] [29] for the complex Lagrangian for spin $3/2$ fields propagating on a (fixed) curved background space-time with symmetric, metric connection employed by Frauendiener, [35][36] that is,

$$L_3^2(\lambda, \bar{\lambda}, \mu, \psi) = \lambda^A \wedge \nabla A - \Sigma^{AB} \wedge (\psi_{ABC} + \epsilon_{C(AB)}^E \lambda^C), \quad (41)$$

with $\lambda_{AA'B'}$ chosen to be the complex conjugate of $\kappa_{A'AB} [37],

$$\lambda_{A(A'B')} = \kappa_{A'(AB)}. \quad (42)$$

Here $\lambda^A$ is a two form defined in terms of the $\Sigma$-basis,

$$\lambda^C = \lambda^{CEF} \Sigma_{EF} + \epsilon^{E(F)} \Sigma_{EF} + \lambda^{CB'C'} \Sigma_{B'C'},
\quad \in (\frac{3}{2}, 0) \oplus (\frac{1}{2}, 0) \oplus (\frac{1}{2}, 1), \quad (43)$$

where $r^F := \frac{2}{3} \lambda^D D^F$ and the field equations arising from the variation of $\psi$ and $\mu$ are

$$\Sigma^{(AB} \wedge \lambda^{C)} = 0, \quad (44)$$

$$\Sigma^{AB} \wedge \lambda_B = 0. \quad (45)$$

Any one of the chiral Lagrangians, [37][14] for gravitation discussed previously and in particular the Nieh-Yan one of Tung, [26],

$$L_{g_3^2} = iD\theta^{AA'} \wedge D\theta_{AA'}, \quad (46)$$

together with $L_3^2(\lambda, \bar{\lambda}, \mu, \psi)$ constitute a chiral Lagrangian for Supergravity [38] with one (N=1) supersymmetry generator; the simplest consistent coupling of a Rarita–Schwinger spin $3/2$ fields to gravity. Equation [44] on its own determines the left-handed two-form as

$$\lambda^C = r^F \Sigma^C F + \lambda^{CEF} \tilde{\Sigma}_{E^F},$$

$$= \theta^{CEF} \wedge \{- \frac{1}{2} \theta_{FE'} - \theta_{F'E'} r^F E^F\},$$

$$:= \theta^{CA'} \wedge \tilde{\kappa}_{A'}. \quad (47)$$

\[14\] We can define a charge for $3/2$ fields by performing a Legendre transformation on chiral Lagrangians using symplectic techniques.
while (45) by eliminating $r^F$ gives the two-form representation of the potential

$$\lambda^A = \lambda^{(A'B')} \Sigma^{A'B'}, \quad (48)$$

thus providing a Dirac form of the Rarita-Schwinger equations, $\nabla \lambda^A = 0$. The contact potentials, $\kappa^A$ of (41) possess the gauge freedom

$$\delta_\nu \kappa^A = \nabla^A \theta_B^B = \nabla^B \bar{\theta}_B^B = -\Sigma^{A'B'} \theta_B^B, \quad (49)$$

so that the Lagrangian, (41) possesses a symmetry if the condition, $\varphi_2^2$ is satisfied on the background space-time. This symmetry is generated by an infinitesimal (real) spinor parameter, $\nu^A$. Denoting the solution space of the Dirac form of the equations for the potentials by $\phi_3^2$, solutions of $\phi_3^2$ can be obtained by taking a symmetrised derivative,

$$\nabla^{B(B^B)} \theta_B = 0, \quad (51)$$

for all Weyl spinors, $\nu \in \varphi_1^2$ in the solution space of the Weyl equation

$$\nabla^{CC'} \nu^C = 0. \quad (52)$$

This is one of the motivation for Penrose’s Twistor program [33], [34]: a Lax-pair of Weyl equations whose integrability conditions are the Rarita-Schwinger equations which in turn have integrability conditions that are Einstein’s vacuum equations [13].

**Half-flat charge from contact form**

The respective Hamiltonian, (see Frauendiener, [35]) for these Noether symmetries that preserve the contact form $\kappa^A$ follows from a Legendre transformation on (41) and is the surface integral,

$$H_\nu(\kappa_A) = \int_{\partial \Sigma} \sigma_\nu.$$

This result is to be contrasted with that flat space expression for the charge obtained by considering the complex potential (see Esposito, [?]), $+A$ defined as

$$+A := \lambda^{A'B'A'B} (B^B \theta^A')^A.$$

With $\psi_{A'B'C'} =: \partial_{A'(A'} \lambda^{A'B'C')}^A$ the helicity ($\frac{3}{2}$) field strength and $\mu^A$ is to be interpreted as the primary part of a dual twistor, $(\mu^A, \gamma_A)$, where

$$\partial_{AA'} \mu^B = i \epsilon_{A'B'} \gamma_A \quad \text{and} \quad \partial_{AA'} \gamma_B = 0. \quad (53)$$

The field equation of the spin potential, $\kappa$ determines $+F$ as self-dual,

$$+F = [\psi_{A'B'C'} \mu^C + i \lambda_{A'B'A} \gamma^A] \Sigma^{A'B'},$$

having an associated electric charge,

$$Q^E = -\int_{\partial \Sigma} +F = -\int_{\partial \Sigma} \{\mu^C \psi_{A'B'C'} + i \lambda_{A'B'A} \gamma^A \} \Sigma^{A'B'}.$$
The imaginary part of the flat space expression representing the electric charge corresponds then to the symplectic expression,

\[ H_\nu(\sigma) = \int_{\partial \Sigma} \lambda_{A A' B'} \gamma^A \Sigma^{A' B'} - \kappa_{A B A'} \Sigma^{A B}, \]

if \( \gamma^A \) is identified with \( \nu^A \) and \( \pi^{A'} \) is identified with \( \bar{\nu}^{A'} \). This agrees with the comparable calculation given by Frauendiener, \[35\] who employed a non-chiral Lagrangian.

**Discussion**

Witten’s, \[5\] covariant formalism was used to explore the symplectic structure of various cohomologous (semi) chiral Einstein-Cartan Lagrangians. The novelty offered here was a heuristic pedagogical introduction of the geometric formalism that made frequent contact to more familiar derivations of Noether conserved currents. The functional derivative of the boundary term from these Lagrangians, being the symplectic potential results from performing an integration by parts in the Variational Principle. The GHS \[7\] push of the Liouville measure, \[11\] on the physical space of trajectories resulting from on-shell variations of any one of the chiral Lagrangians, \[37\] could be used to explore chiral aspects related to the CMBR fine-tuning issue as Carrol proposes, \[8\]. The quasi-local Noether charge, as a surface Boundary term was used to define distinct energy-momentum pseudo-tensor-valued forms. Their Noether surface charge densities on the boundary will need to be appropriately fixed in order that distinct solutions from the equations of motion can be determined.

Applying the spinor-valued symplectic formalism we revealed the gauge freedom afforded to the contact one form for a Chiral (left-handed) Rarita-Schwinger Lagrangian. This related the associated Noether symmetries implicitly to the on-shell satisfaction of the linearised Einstein’s Vacuum equations and a relation to a Weyl equation Lax Pair to the Twistor equation was made. The charge for a complex half-flat space was derived and on the way a chiral N=1 Lagrangian for supergravity was constructed.

**Notes**

\[1\]The Bel–Robinson super-energy tensor is the gravitational equivalent to the symmetric energy–momentum tensor of the electromagnetic field

\[ T_{abcd} = \frac{1}{4} (C_{eabf} C^{e f} + C^*_{eabf} C^{*e f}), \]

where \( C^*_{abcd} = \frac{1}{2} \eta_{abc f} C^{e f}_{\ cd} \) is the dual of the Weyl tensor. As such it is overall symmetric, tracefree, and is covariantly conserved in vacuum. Its form is determined soley as a consequence of the Bianchi identities, written in two components spinors as,

\[ T_{abed} \leftrightarrow \Psi_{ABCD} \tilde{\Psi}_{A'B'C'D'}, \]

and it has been proposed as the covariant (if observer-dependent) object to describe the Entropy of the gravitational field in [Gravitational Entropy Proposal](Gravitational Entropy Proposal)

The Bel-Robinson tensor and its divergence arise as a consequence of the Integability conditions of Einstein’s equations, that is the Bianchi identities and their contractions. The components of the Bel–Robinson tensor, \( T^a_{bcd} \) have dimension \( t^{-4} \) so that \( \frac{1}{2 \pi} T^{a}_{bcd} \) for \( \beta = \frac{8 \pi G}{c^4} \) has dimensions of the square of energy-momentum.
Consider a transformation that combines a constant spacetime translation together with a particular gauge transformation, an ‘improved translation’ that is gauge invariant. The action of a constant translation on a 1-form $A_\mu$ is gauge invariant as $F$ only appears

\[
\delta_0 A_\mu = -\epsilon^\mu(x)\partial_\mu A_\mu, \\
\delta A_\mu = -\epsilon^\alpha \partial_\alpha A_\mu + \partial_\mu(\epsilon^\alpha A_\alpha) = F_{\mu\nu}\epsilon^\nu.
\]

This tensor is gauge invariant (since $F$ is) and has zero trace (associated to the scale invariance of the classical theory). Consider now the variation plus a gauge transformation

\[
\delta A_\mu(x) = -\xi^\alpha \partial_\alpha A_\mu - \partial_\nu \xi^\nu A_\mu + \partial_\mu(\xi^\nu A_\nu), \\
= F_{\mu\nu}\xi^\nu(x).
\]

the gauge transformation cancels the term with derivatives of $\xi^\mu$ and at the same time all derivatives of $A_\mu$ appear through $F_{\mu\nu}$. This variation is called an ‘improved diffeomorphism’. When we add an extra piece to $\delta A_\mu$, this should in principle modify the conserved current and it’s associated charge. This does not occur because gauge symmetries are not generated by physical charges but instead by constraints which vanish on-shell. Thus the “charge” associated to a gauge symmetry is always zero and adding an extra gauge transformation to $\delta A_\mu$ does not alter the conserved current. We distinguish the for Maxwell we have

\[
\delta_{\text{gauge}} A_\mu(x) := A'_\mu(x) - A_\mu(x) = D_\mu(x), \\
\delta_{\text{diff}} A_\mu(x) := A'_\mu(x) - A_\mu(x) = \xi^\alpha A_{\mu\nu} - \xi^\nu(x)A_\nu(x).
\]

where all fields and parameters are evaluated at the same point, $x$.

Maxwell’s theory possesses the 15 dimensional conformal group as its Noether symmetry. It is not invariant for arbitrary vectors field $\xi^\mu(x)$ under $\delta A_\mu(x) = -\xi A_\mu(x)$, as

\[
\delta A_\mu(x) = F_{\mu\nu}\xi^\nu(x).
\]

It is though invariant under transformations that belong to the conformal group: those variations such that $\xi^\mu(x)$ satisfies the conformal killing equations,

\[
\xi_{\mu,\nu} + \xi_{\nu,\mu} = \frac{1}{2}\eta_{\mu\nu}\xi^\alpha.
\]

That is to say that the transformation leaves Maxwell’s Lagrangian invariant (up to a boundary term), if the vector $\xi^\mu$ satisfies. We want to write the variation of the Lagrangian as $\delta_\xi L = f(\xi) + \partial_\alpha \theta^\alpha$, for some function $f$. As we wish to impose the conformality restriction over the kind of transformations of coordinates but not on the dynamical fields we require this to depend on $\xi$ but not on the field $A_\mu$. Given $\mathcal{L} = \frac{1}{4}F^2$

\[
\delta_\xi L = F^{\mu\nu}\partial_\mu(\xi^\rho F_{\nu\rho}), \\
= -\frac{1}{2}F^{\mu\nu}F^\rho_{\nu}(\partial_\rho \xi_\mu + \partial_\mu \xi_\rho - \frac{1}{2}\eta_{\mu\rho}\partial_\alpha \xi^\alpha) - \partial_\rho(\xi^\rho L),
\]

Thus, the transformation $\delta x^\mu = \xi^\mu$ is a symmetry of Maxwell’s action provided that the diffeomorphism satisfies the conformal Killing equation so that

\[
\delta_\xi L = d(\xi_\mu L)
\]

Thus, the variation $\delta x^\mu = \xi^\mu(x)$. Invariance under any general $\xi^\mu$ is the symmetry of General Relativity (GR) so when Maxwell’s theory is coupled to a dynamical metric it becomes invariant under such general diffeomorphisms. Having a time-like Killing vector would normally allow one to define a conserved energy that is inferred solely from the material stress-energy and that is not coordinate-dependent. But as we have seen, the symmetries of some actions for Einsteins equations give rise to
only pseudo-tensor conservation laws, whether or not there are Killing vectors. As such the energies and momenta always being conserved in GR is as a consequence of Einstein’s equations, and the triviality of its integrability equation, the Bianchi identity. It is sometimes argued that you cannot find a time-like Killing vector field for energy to be conserved because the metric in Einstein-Hilbert, GR has Euler-Lagrange equations. Noether’s theorem assumes that every dynamical field of the Variational Principle has a corresponding Euler-Lagrange equations, δL. Those fields that lack an Euler-Lagrange equation (linearised with flat metric) have associated symmetries instead. The more of such absolute fields, the fewer vector fields that leave the Lagrangian density (quasi-)invariant. In GR there are no such fields, so every vector field is a symmetry of the Lagrangian.

Goldberg’s Lagrangian \( \mathcal{L}_{CG} \) may also be viewed as a teleparallel version of \( \mathcal{L}_{SC} \) with the curvature of the \( sl(2, \mathbb{C}) \) connection decomposing according to the teleparallel condition,

\[
\mathcal{F}^{AB} \delta A^B + \mathcal{F}^{A'B'} \delta A_B = 0,
\]

both yielding vanishing torsion as the chiral breaking connection field equation is equated to zero. When coupling chiral spinor matter to the theory, a consistent set of equations fails to result if the quadratic torsion term is not added.

Consider the conserved quantities associated to the two cases of charging and then discharging a parallel plate capacitor. There are two scenarios, (1) after charging, disconnecting the battery and measuring the work required to remove the dielectric, versus (2) leave the battery connected, allow the current to flow as the capacitor discharges. The amount of work (our conserved energy quantity) required is not the same in these two cases. In the second scenario we would measure the work with the potential fixed as in Dirichelet boundary fixing condition. While the "symmetric-Hilbert-like" Nester energy momentum Hamiltonian \( H(\sigma) \) captures the first scenario, the "canonical electromagnetic" Hamiltonian \( H(\phi) \) reflects the latter so-called Neumann conditions,

\[
H(\sigma) = \int \left( \frac{1}{2} (E^2 + B^2) + \phi \nabla \cdot E \right) d^3 x,
\]

\[
H(\phi) = \int \left( \frac{1}{2} (E^2 - B^2) + \phi \nabla \cdot E \right) d^3 x = H(\sigma) - \phi E \cdot n dS
\]

\( H(\sigma) \) comprises both the energy density and Lagrange multiplier gauge term that delivers Gauss’s law \( \nabla \cdot E \). According to Neumann, it includes thus a boundary term that will vanish if we fix on the boundary, the normal component of the electric field, \( E \),

\[
\delta H(\sigma) = \int \phi \delta (E \cdot n) dS.
\]

That is by fixing \( \sigma \), the surface charge density on the boundary.

\( H(\phi) \) derived from a cohomologous Lagrangian scalar density differing by a boundary term, has a variation with different boundary term that according to Dirichlet, vanishes if we fix the scalar potential field on the boundary. Quasi-local charge definitions differing by boundary terms will thus be expected to have different boundary condition specifications being either of Neumann-field or Dirichelet-potential fixing form and thus possess distinct physical interpretations.

References

*Volume elements of spacetime and a quartet of scalar fields,*


*A Natural Measure on the set of all Universes* Nucl. Phys.,(1984), 281,736.


[9] Pons J and Salisbury D. *The issue of time in generally covariant theories and the Komar-Bergmann approach to observables in general relativity,* 


[12] Banados, M. and Reyes, I. *A short review on Noether’s theorems, gauge symmetries and boundary terms*


[15] Nester J. 
http://ymsc.tsinghua.edu.cn/upload/news_201412316849.pdf


[17] Krasnov K., Plebanski Formulation of General Relativity: A Practical Introduction, 


[26] Tung R, Jacobson T Spinor one-forms as gravitational potentials, 


[31] Penrose, R. (1991). Twistor spin $\frac{3}{2}$ charges, Gravitation and Modern Cosmology


[34] Penrose R., Palatial twistor theory and the twistor googly problem,

http://rsta.royalsocietypublishing.org/content/373/2047/20140237#sec-7


[38] Mielke E. and Macias A, Chiral supergravity and anomalies,
